Indistinguishability Obfuscation via Mathematical Proofs of Equivalence

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Program Obfuscation

```
function main() {
  console.log('hello, world');
}
main()
```

```
function _0x19e6(_0x4d301f,_0xcaab53){var _0x3a4e72=_0x3a4e();return
_0x19e6=function(_0x19e691,_0x5809f0){_0x19e691=_0x19e691-0x14e;var
_0x16ee0b=_0x3a4e72[_0x19e691];return
_0x16ee0b;},_0x19e6(_0x4d301f,_0xcaab53);}function _0x3a4e(){var _0x3f0a9d=
['log','199381NCGrSa','2328491tAiNSg','18mVqyqS','4cVQTsk','6PuGzwR','107410
32WsiTVO','104321yYIIVM','370911DTLqdw','10uRQffV','2024504eEkwnt','114dOcOh
j','hello,\x20world','2634710IatlOd'];_0x3a4e=function(){return
_0x3f0a9d;};return _0x3a4e();}(function(_0x3d9e47,_0x360e03){var
_0x3afd0b=_0x19e6,_0x2928d3=_0x3d9e47();while(!![]){try{var _0x33cc3a=-
parseInt(_0x3afd0b(0x15a))/0x1*(-parseInt(_0x3afd0b(0x158))/0x2)+-
parseInt(_0x3afd0b(0x15b))/0x3*(-parseInt(_0x3afd0b(0x157))/0x4)+-
parseInt(_0x3afd0b(0x152))/0x5+parseInt(_0x3afd0b(0x150))/0x6*
(parseInt(_0x3afd0b(0x154))/0x7)+-parseInt(_0x3afd0b(0x14f))/0x8*(-
parseInt(_0x3afd0b(0x156))/0x9)+parseInt(_0x3afd0b(0x14e))/0xa*
(parseInt(_0x3afd0b(0x155))/0xb)+-
parseInt(_0x3afd0b(0x159))/0xc;if(_0x33cc3a===_0x360e03)break;else
_0x2928d3['push'](_0x2928d3['shift']());}catch(_0x437e27){_0x2928d3['push']
(_0x2928d3['shift']());}}}(_0x3a4e,0x42c94));function main(){var
_0x29ace6=_0x19e6;console[_0x29ace6(0x153)](_0x29ace6(0x151));}main();
```

Indistinguishability Obfuscation (iO)

C: Program (Circuit/Turing Machine)

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iO: obfuscator $iO(1^\lambda, C) \rightarrow C'$, ($\lambda$: Security parameter)

- **Preserve Functionality:**

$$\forall\, x, C'(x) = C(x)$$

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- **Preserve Functionality:**

$$\forall\, x, C'(x) = C(x)$$

- **Indistinguishability Security:** for any C_0, C_1

$$\forall\, x\ C_0(x) = C_1(x), \quad iO(1^\lambda, C_0) \approx_c iO(1^\lambda, C_1)$$

Indistinguishability Security, as a **Game**

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n.u.
Probabilistic
Poly.-time

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$C_0, C_1 \longrightarrow$

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$\xrightarrow{C_0, C_1}$

$b \leftarrow \{0,1\}$

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C_0, C_1 →

← $iO(1^\lambda, C_b)$

$b \leftarrow \{0,1\}$

Indistinguishability Security, as a **Game**

n.u.
Probabilistic
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$$\xrightarrow{\color{red}{C_0}, \color{blue}{C_1}}$$

$$\xleftarrow{iO(1^\lambda, C_b)}$$

$$\xrightarrow{b' \in \{0,1\}}$$

$b \leftarrow \{0,1\}$

Indistinguishability Security, as a **Game**

C_0, C_1 →

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$b \leftarrow \{0,1\}$

← $iO(1^\lambda, C_b)$

$b' \in \{0,1\}$ →

If $\forall \boldsymbol{x}\ \boldsymbol{C_0}(\boldsymbol{x}) = \boldsymbol{C_1}(\boldsymbol{x}) \wedge$
$b = b'$, adversary wins

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n.u.
Probabilistic
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$C_0, C_1 \longrightarrow$

$iO(1^\lambda, C_b) \longleftarrow$

$b' \in \{0,1\} \longrightarrow$

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If $\forall \boldsymbol{x}\ \boldsymbol{C_0(x) = C_1(x)} \wedge$
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$$\boxed{\Pr[\text{Adversary } wins] \leq \text{negl}\ (\lambda)}$$

Indistinguishability Security, as a **Game**

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C_0, C_1 →

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If $\forall \boldsymbol{x}\ \boldsymbol{C_0(x) = C_1(x)} \wedge$
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<u>Non-Falsifiability</u>
The challenger (judger) is inefficient.
($2^{|x|}$-time)

Good Assumptions are **Falsifiable**

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<u>Example: Factoring</u>

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$p, q \leftarrow Primes$

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Falsifiable

The challenger is poly-time

Base iO on Good Assumptions?

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- A long line of works:

  [Garg-Gentry-Halevi-Raykova-Sahai-Waters’13][Pass-Seth-Telang’14]
  [Gentry-Lewko-Sahai-Waters’15][Ananth-Jain’15][Bitansky-Vaikuntanathan’15]
  [Lin’16][Lin-Vaikuntanathan’16][Lin-Pass-Karn Seth-Telang’16]
  [Garg-Miles-Mukherjee-Sahai-Srinivasan-Zhandry’16][Ananth-Sahai’17][Lin’17]
  [Lin-Tessaro’17][Agrawal’19][Jain-Lin-Matt-Sahai’19][Brakerski-Dottling-Malavolta’20]…

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- **iO from Well-Founded Assumptions** [Jain-Lin-Sahai'20]
  Based on **Sub-exponential Security** of Learning with Errors, and Learning Parity with Noise and more…

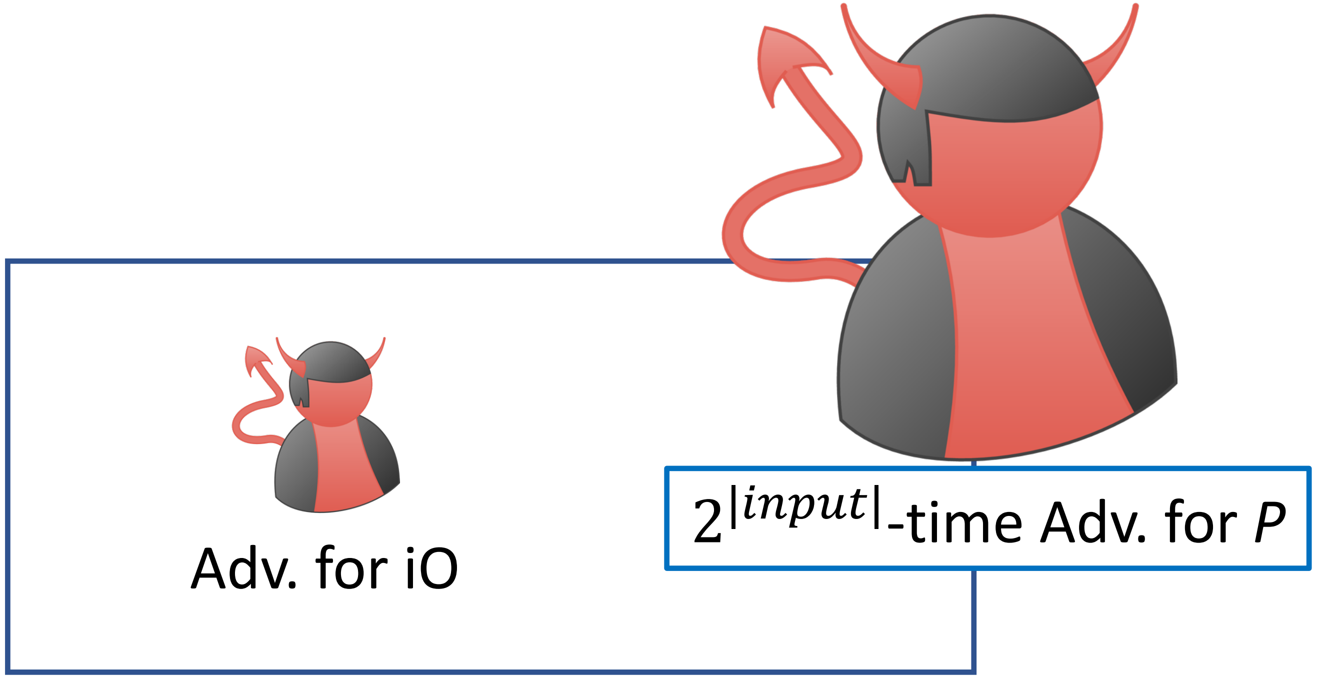
Sub-exponential Security

Of an Assumption P

For any adversary that runs in 2^{λ^c}-time ($0 < c < 1$),
it can only break the Assumption P of size λ with negligible probability.

(P=Learning with Errors, Learning party with Noise, …)

$2^{|input|}$-Loss in Reduction



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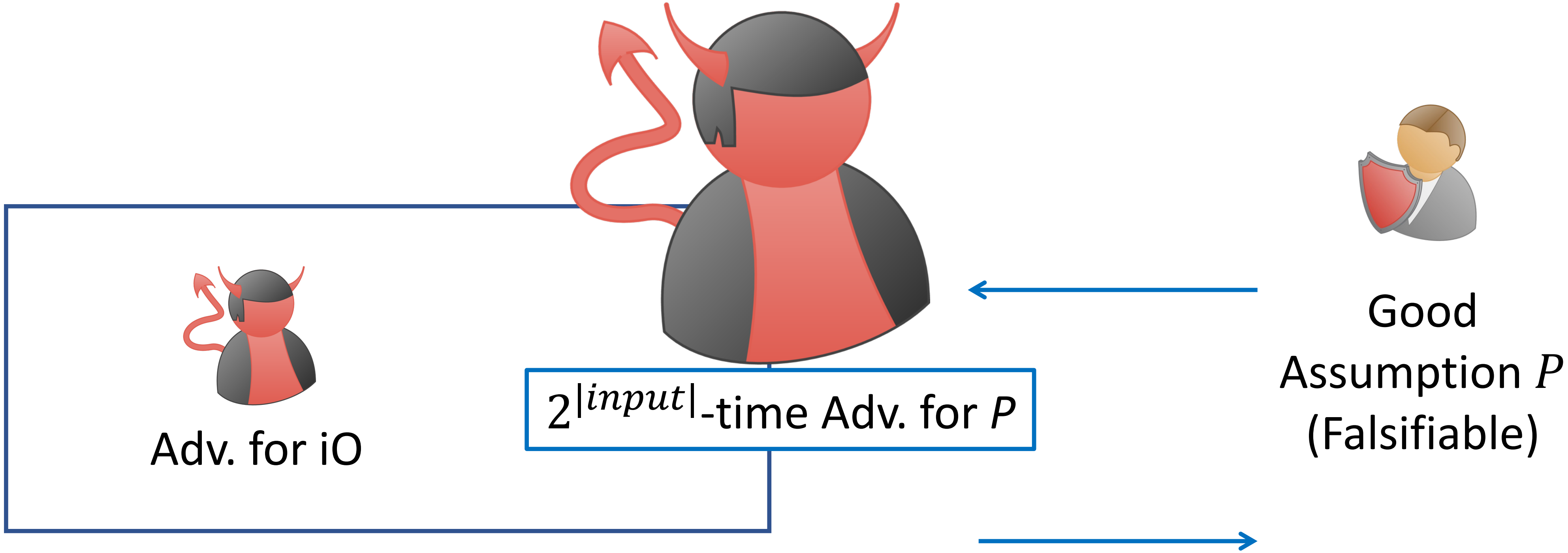

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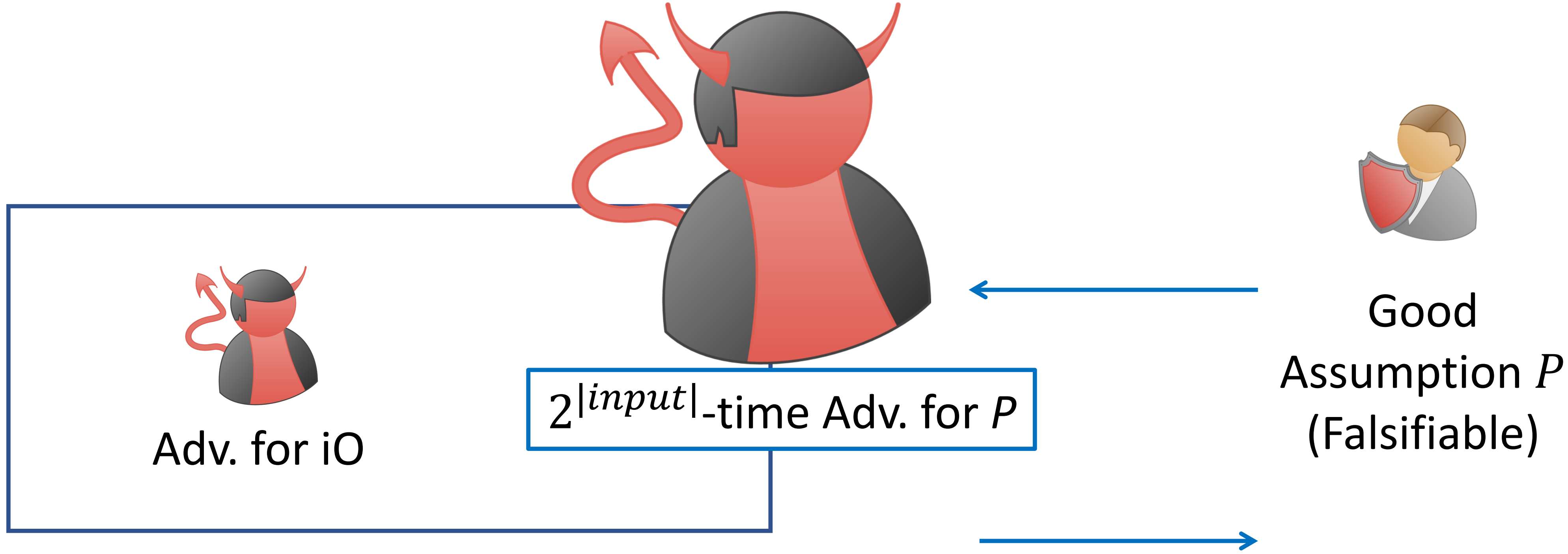

Assume 2^{λ^c}-Security & set $2^{\lambda^c} > 2^{|input|}$

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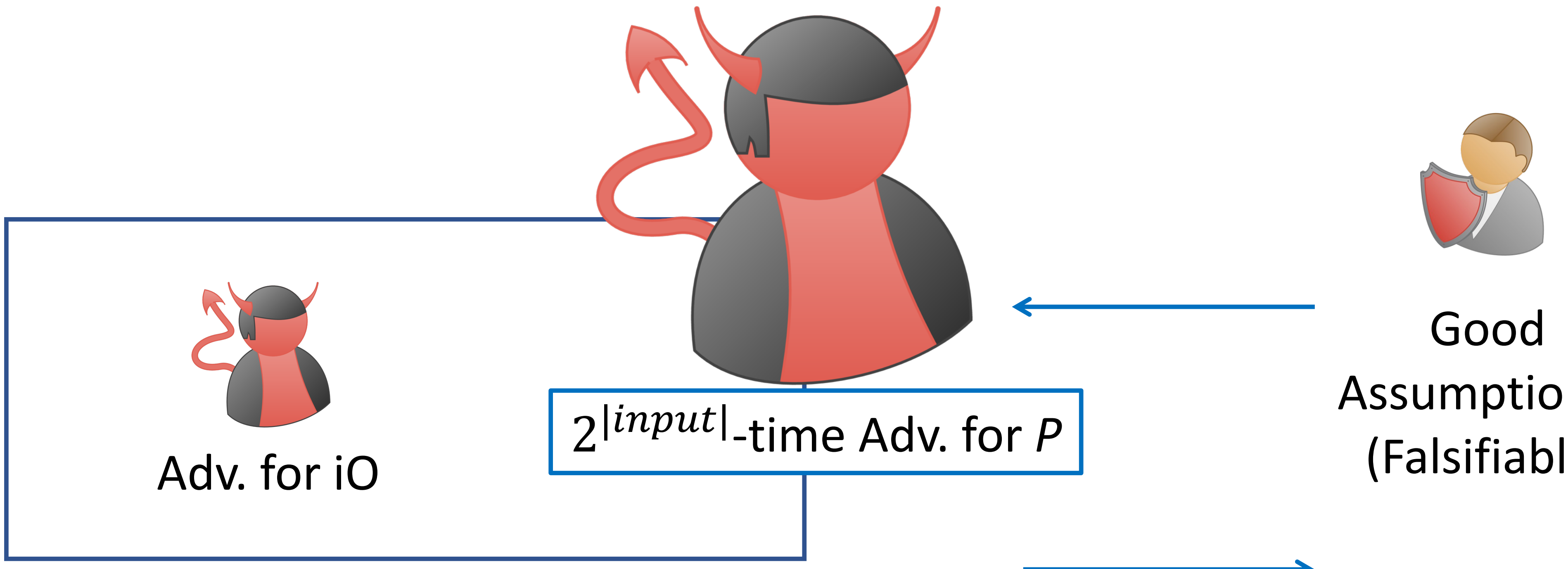

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$|\boldsymbol{input}| < \boldsymbol{\lambda^c}$

$2^{|input|}$-Security Loss is Bad

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iO for Turing Machines: M: a Turing Machine, $iO(1^\lambda, M) \to M'$

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Ideal: M' works for **unbounded input-length**

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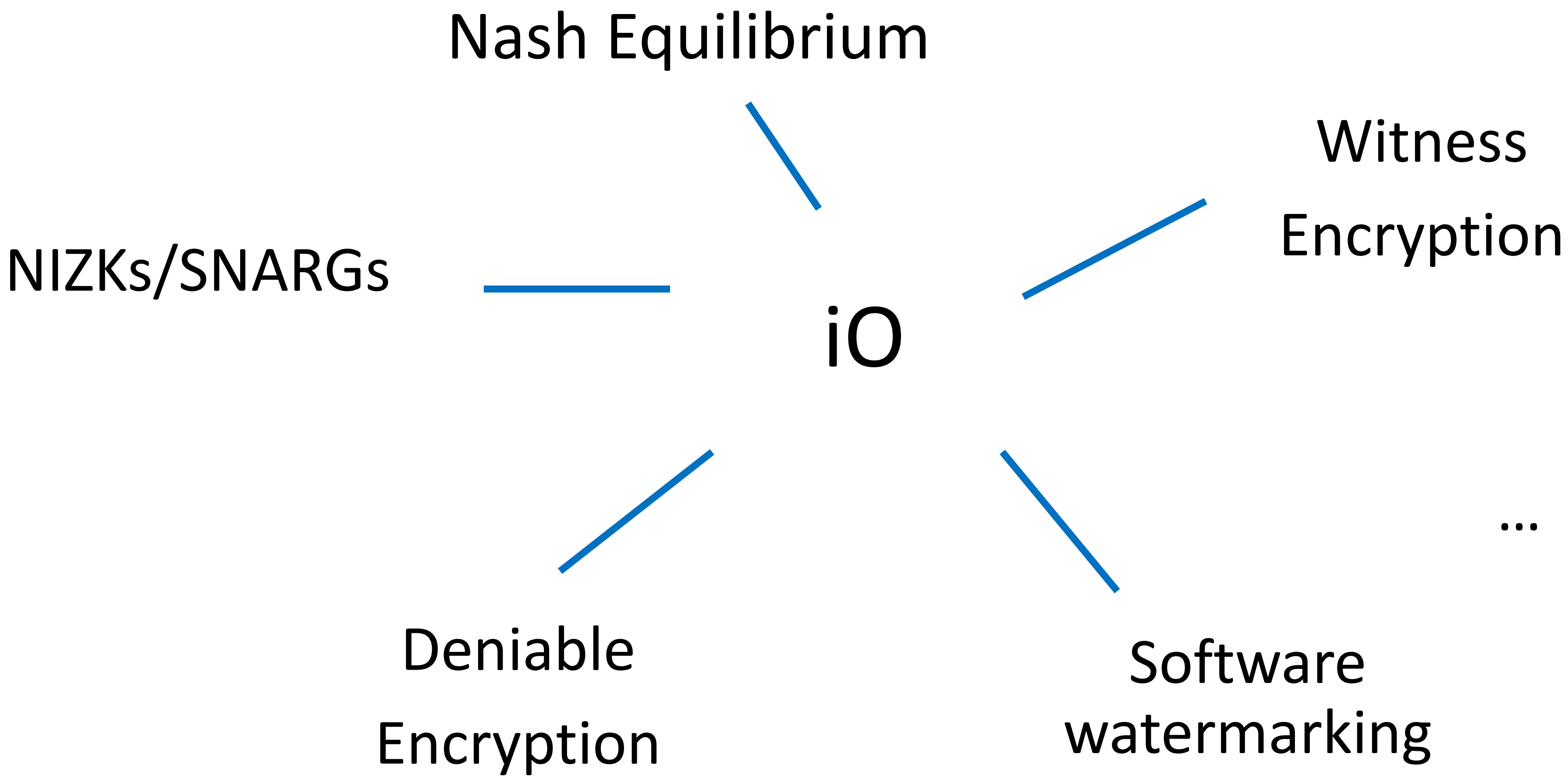
Ideal: M' works for **unbounded input-length**

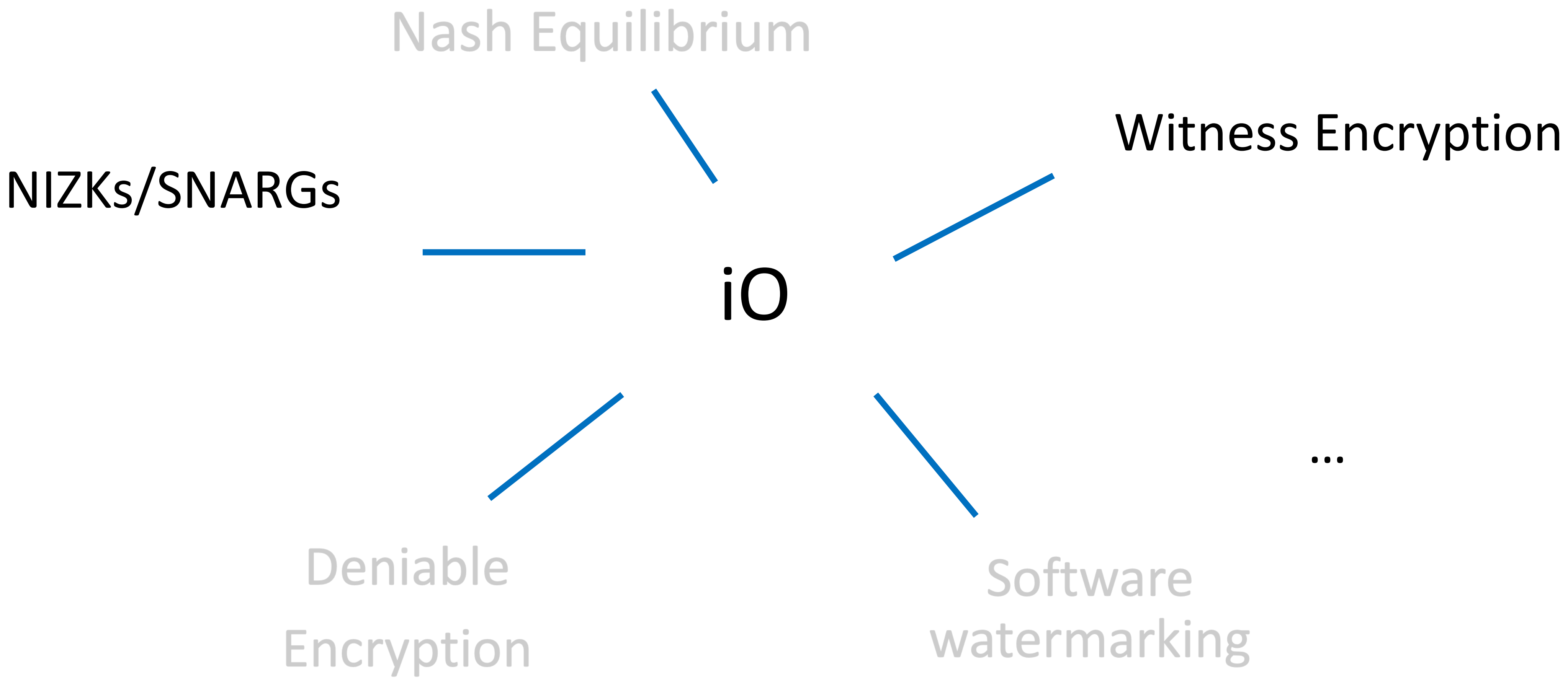
Reality:
Input length is **a-priori bounded (since** $|input| < \lambda^c$**)**

[Bitansky-Garg-Lin-Pass-Telang'15][Canetti-Holmgren-Jain-Vaikuntanathan'15][Koppula-Lewko-Waters'15]…

iO: the "Central Hub" [Sahai-Waters'13]



$2^{|input|}$-Security Loss "Spreads"



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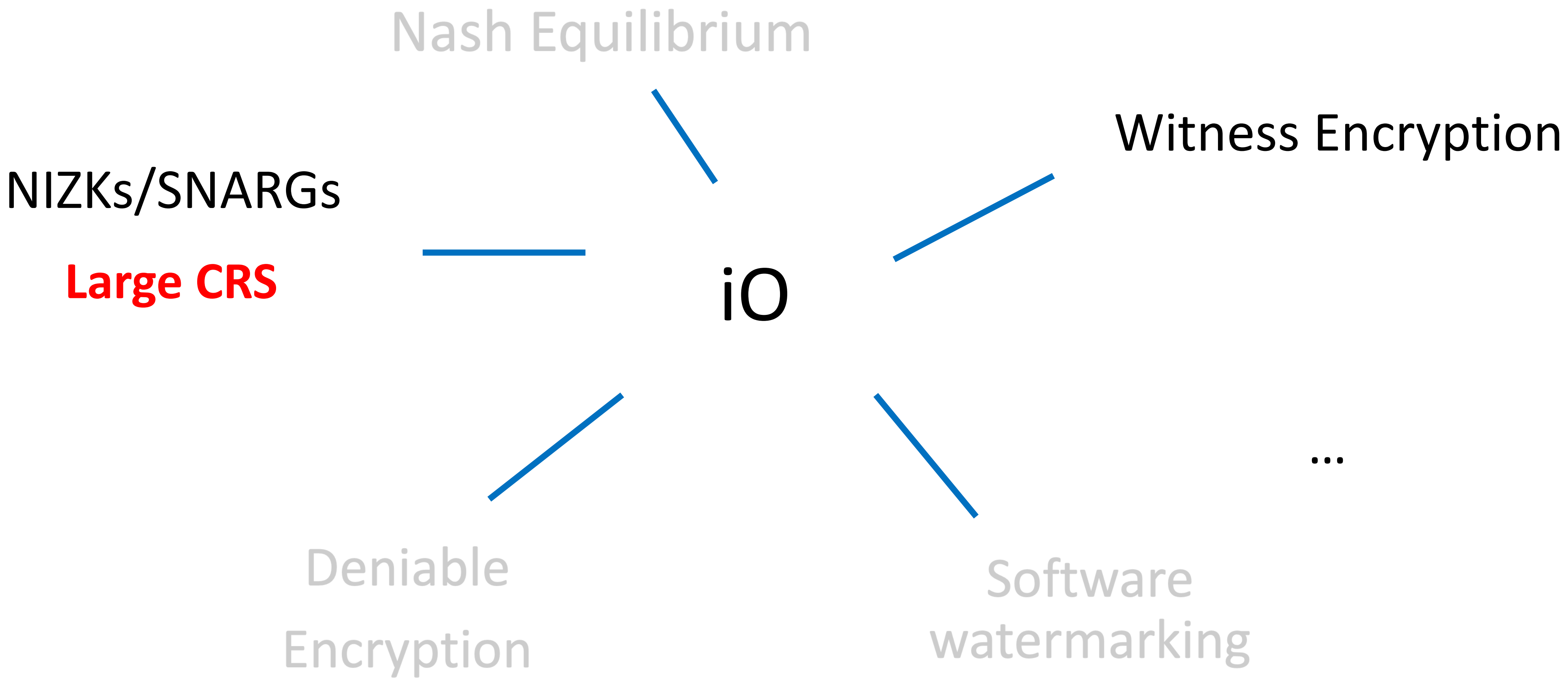

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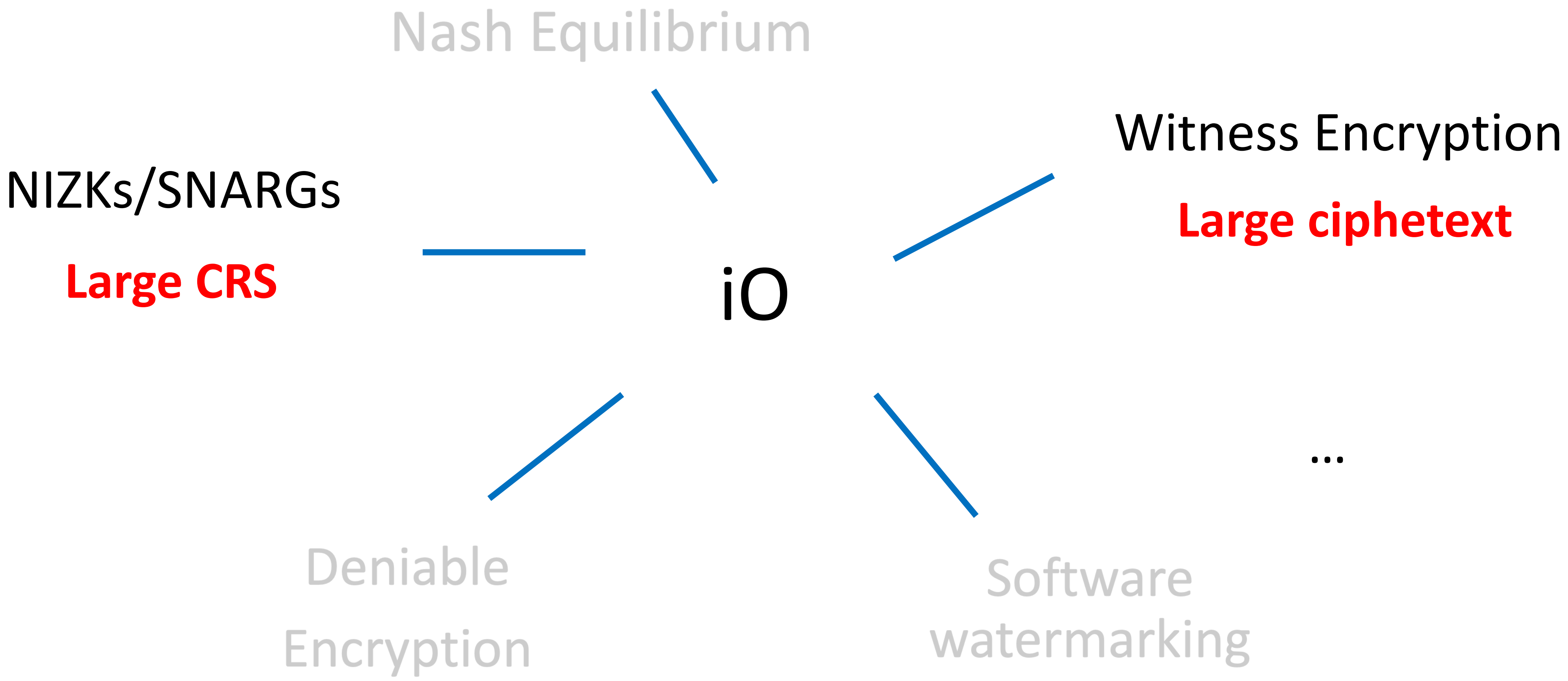
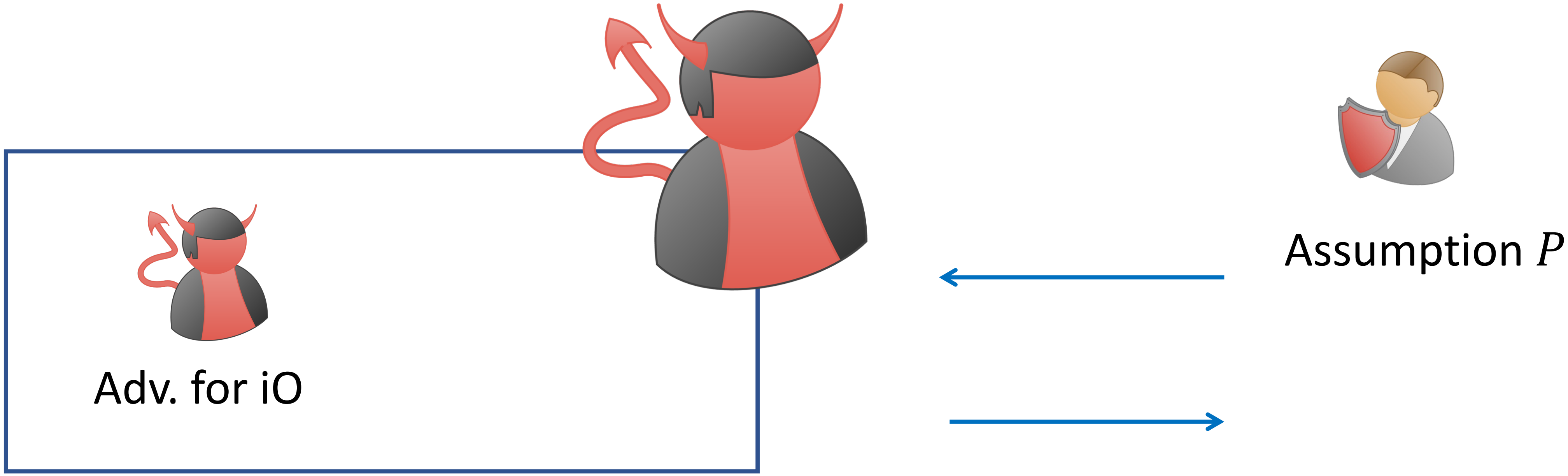


Question: Can we build iO with a security loss ***independent*** of the input length?

Is $2^{|input|}$-Loss Inherent? (folklore)



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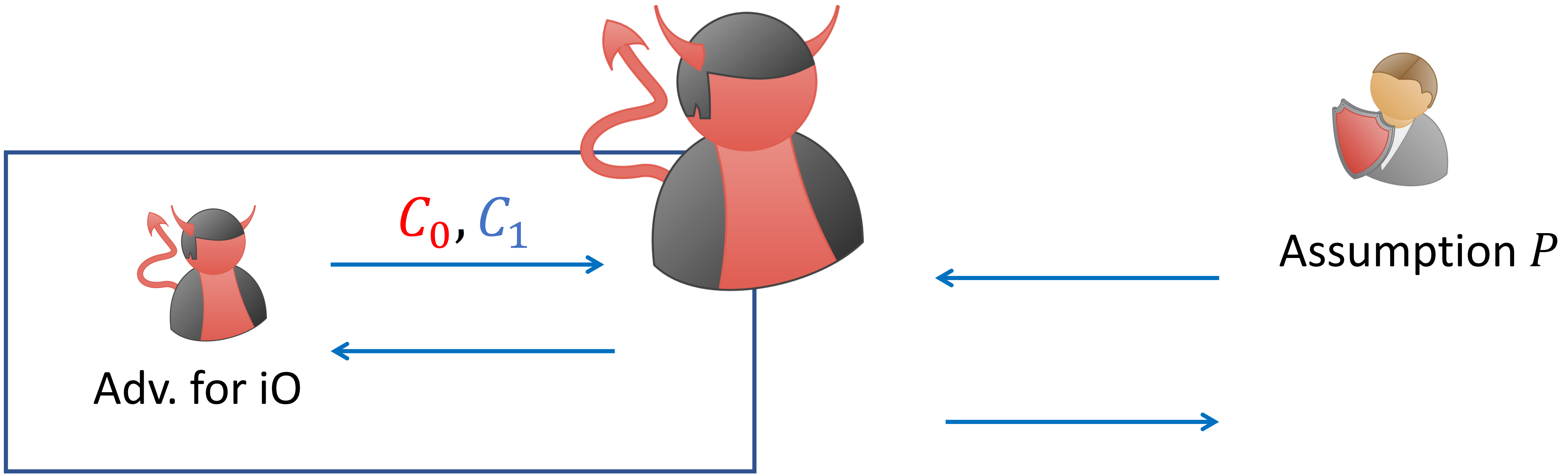

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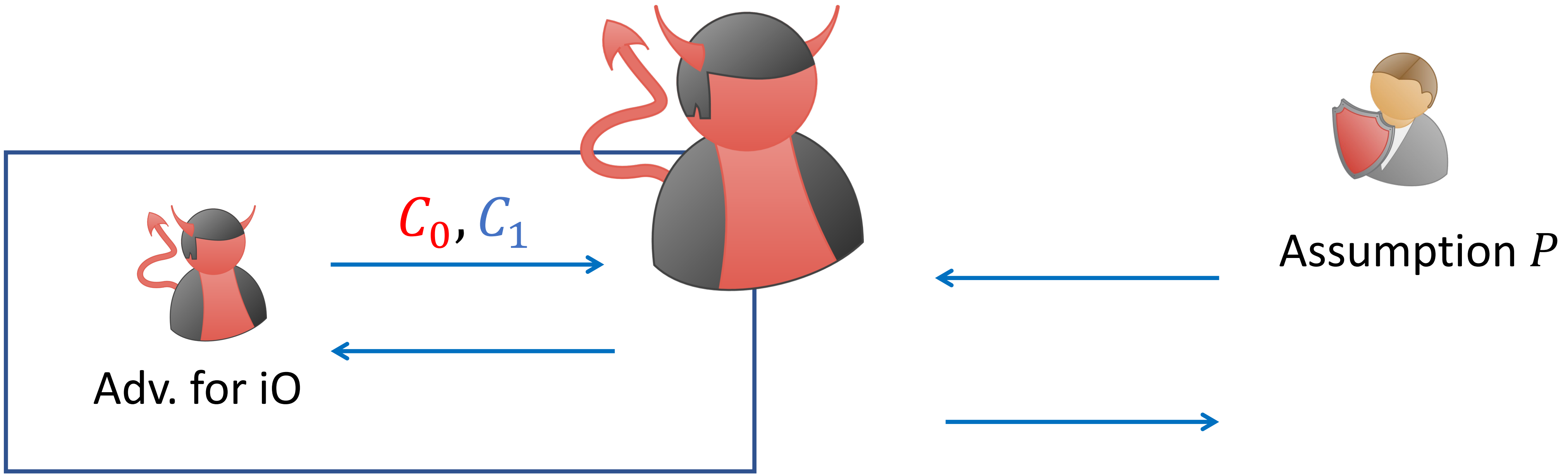

If $\forall x\ C_0(x) = C_1(x)$, then reduction break P.

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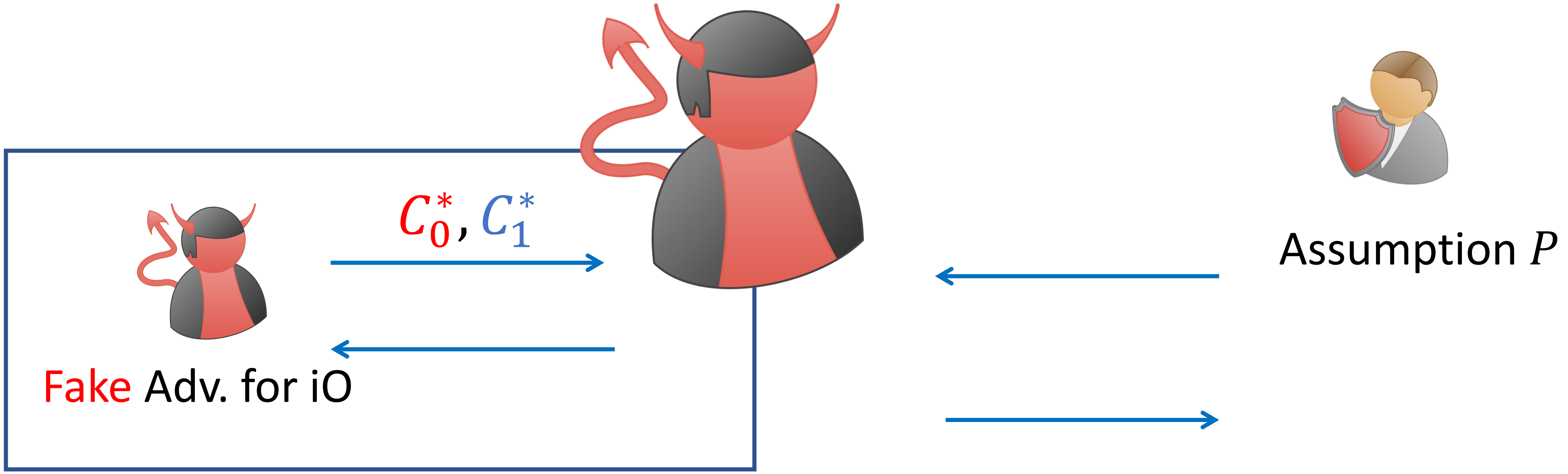

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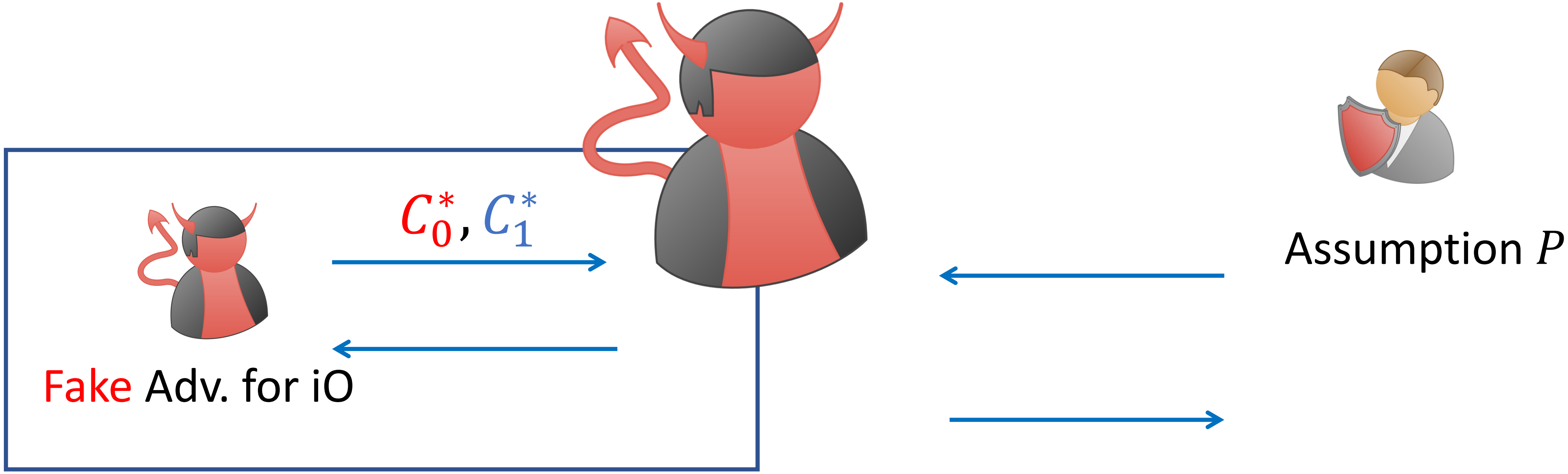

If C_0^*, C_1^* differ at some x^*, then reduction **shouldn't** break P.

Otherwise, P is broken unconditionally.

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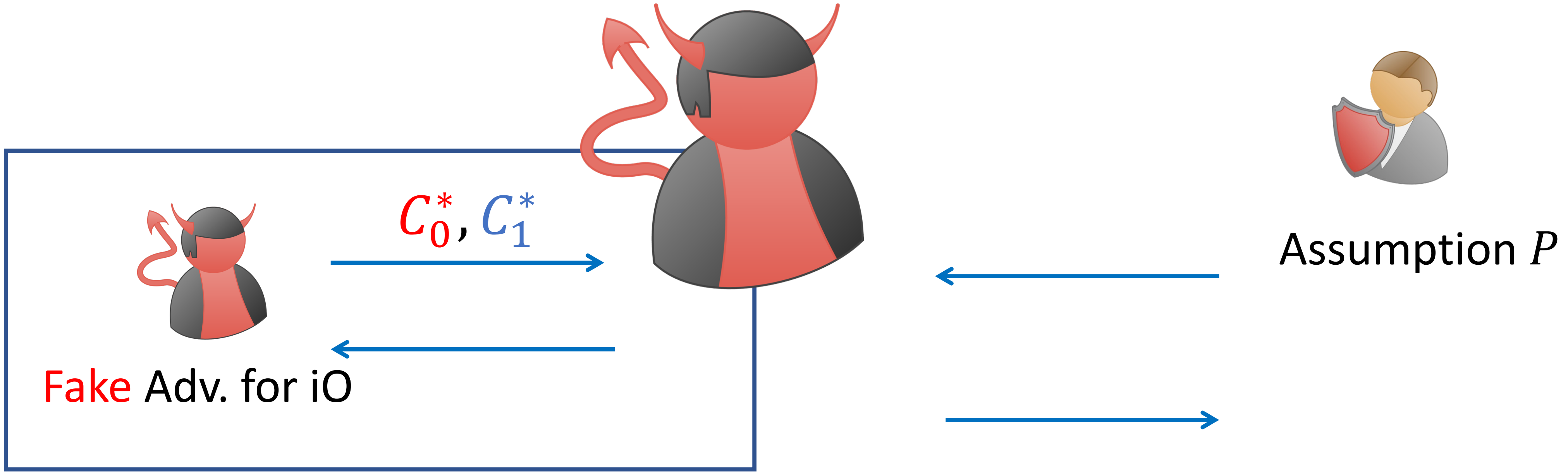

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Otherwise, P is broken unconditionally.

Reduction can't tell, unless it checks at x^*

Broader Perspective

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Such non-falsifiability appears in many other places in crypto

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Example: Non-Interactive Proofs (for $L \in NP$)

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"$x \in L$"

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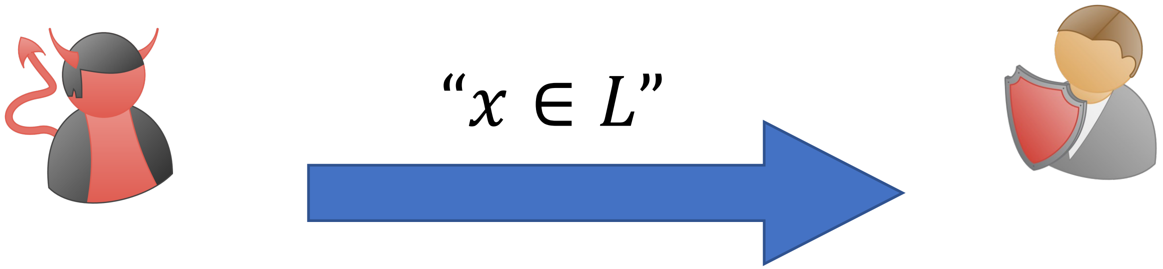

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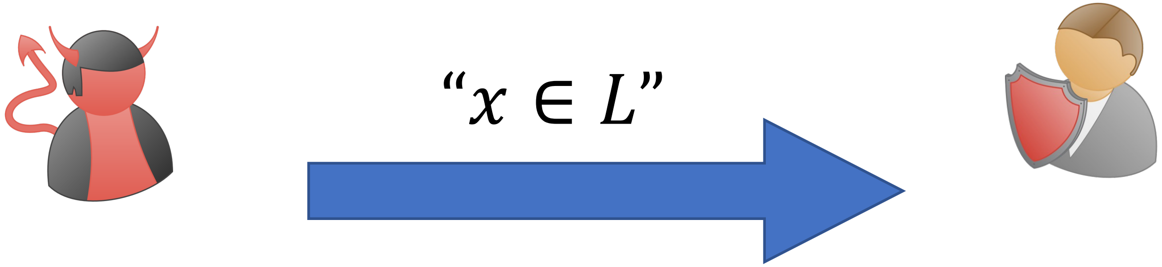
<u>**Example:**</u> **Non-Interactive Proofs** (for $L \in NP$)



Soundness: If $x \notin L$, any cheating proof should be rejected

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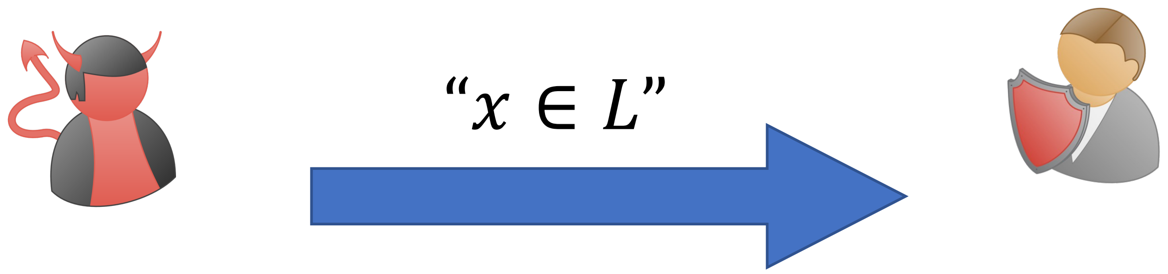

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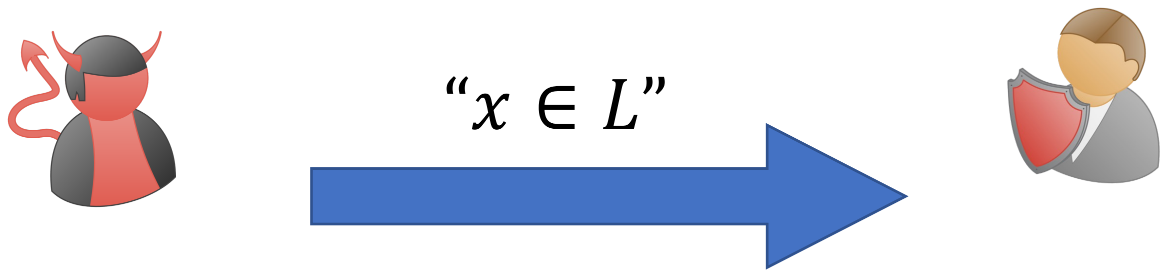

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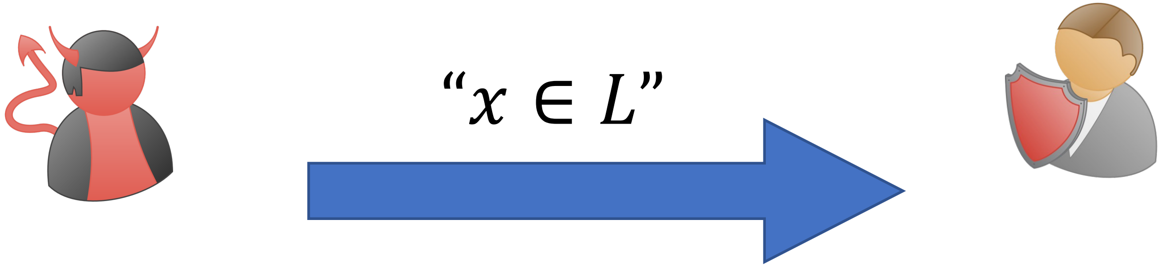

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[Gentry-Wichs’10] impossibility for SNARGs

iO

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"$\forall x\ C_0(x) = C_1(x)$" can be decided in **P**
[Garg-Pandey-Srinivasan'16, Garg-Srinivasan'16, Garg-Pandey-Srinivasan-Zhandry'17]
[Liu-Zhandry'17]

iO

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This Work:

Leverage **math. proofs** of "$\forall x\ C_0(x) = C_1(x)$" to avoid the $2^{|x|}$-time check

Why Such Math. Proofs Exist?

When iO is used in the security proof of other applications:

…

- Construct C_0, C_1
- **Write a math. proof** for $\forall\, x\; C_0(x) = C_1(x)$
- Apply iO security to derive $iO(C_0) \approx_c iO(C_1)$

…

The proof must be "**short**" (length $\ll 2^{|x|}$)

Otherwise, we (human brain) can't understand it.

Our Results I (for Propositional Logic)

$\mathcal{O}$ with security loss independent of $|input|$ for any ckts $\{C_\lambda^1\}_\lambda, \{C_\lambda^2\}_\lambda$ where $C_\lambda^1(x) \leftrightarrow C_\lambda^2(x)$ have **poly-size proofs** in *Extended Frege systems.*

Extended Frege System ($\mathcal{EF}$)

- **Variables**: $p, q, r, \ldots$
- **Formulas**: $p \to r, p \wedge q, \neg p, \ldots$
- **Axioms**:

$$p \to (q \to p)$$
$$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$$
$$p \to \neg\neg p$$

- **Inference Rule:**

$$p, p \to q \vdash q$$

- **Extension Rule:**

$$e \leftrightarrow \phi$$

(assign a new variable e to an existing formula ϕ)

Our Results II (for Cook's Theory PV)

iO for any ***unbounded-input*** Turing machines M_1, M_2,
with $\vdash_{PV} M_1(x) = M_2(x)$.

Assumptions: sub-exponential security of LWE & iO for circuits.

Cook's Theory *PV* [Cook'75] *"Poly-time Reasoning"*

Terms: $f(x), g(x), h(x_1, x_2), \ldots$

Lines are Equations: $f(x) = h(x),\ f\big(x_1, g(x_2)\big) = h(x_1, x_2), \ldots$

Allow definition of any polynomial-time functions, e.g.

- Arithmetic: $+, -, \times, \div, \leq, <, \lfloor\cdot\rfloor, mod, \ldots$
- Logic Symbols: $\rightarrow, \neg, \wedge, \ldots$

Relation Between *PV* and $\mathcal{EF}$

Propositional Translation
[Cook'75]

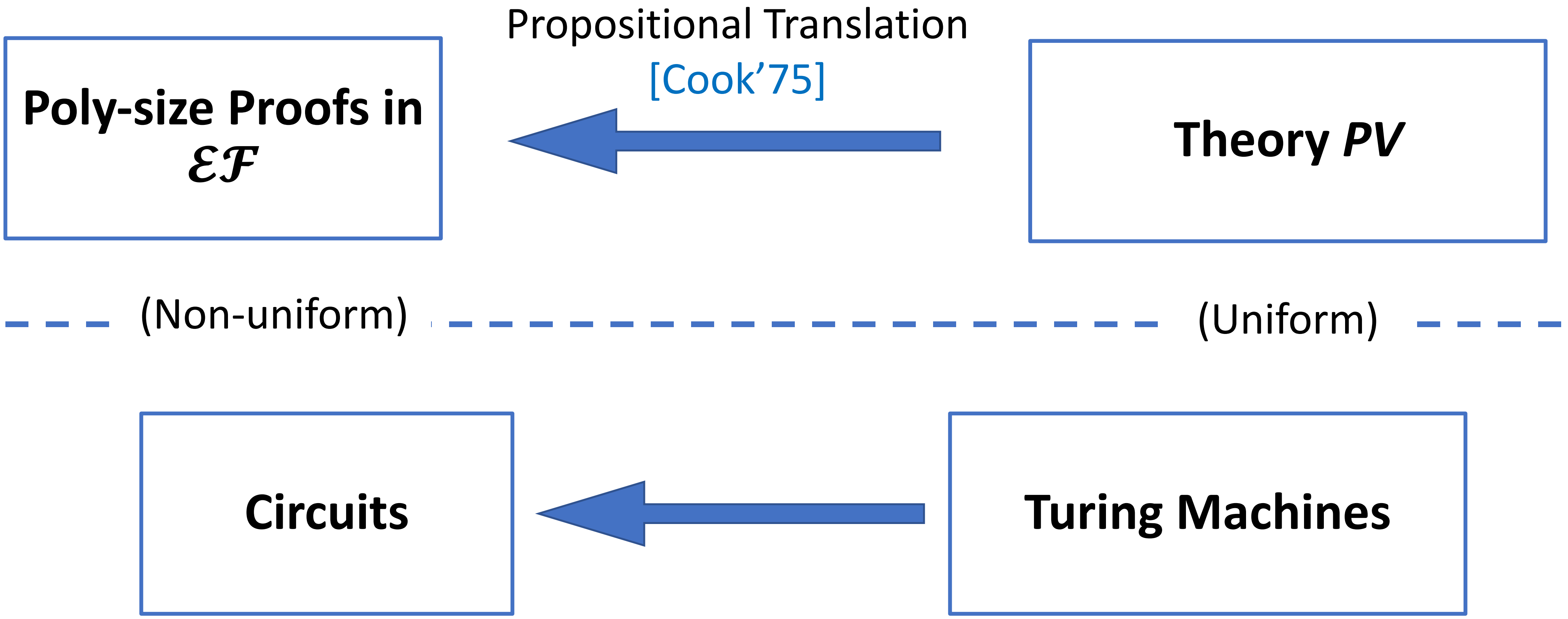
Poly-size Proofs in $\boldsymbol{\mathcal{EF}}$

(Non-uniform)

Theory *PV*

(Uniform)

Relation Between *PV* and $\mathcal{EF}$



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This work:

- Many crypto algorithms are “natural”, e.g.

  ElGamal Encryption,

  Regev’s Encryption

  Commitments,

  Puncturable PRFs

Limitation of PV (Assuming Factoring is hard)

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- Correctness of the algorithm that decides Primes

(Both from Witnessing Theorem)

How to leverage math. proofs?

(An overview)

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<u>Example:</u> Proof of $A \to A$ in *EF*

1. $A \to ((B \to A) \to A)$ (instance of (A1))
2. $(A \to ((B \to A) \to A)) \to ((A \to (B \to A)) \to (A \to A))$ (instance of (A2))
3. $(A \to (B \to A)) \to (A \to A)$ (from (1) and (2) by modus ponens)
4. $A \to (B \to A)$ (instance of (A1))
5. $A \to A$ (from (4) and (3) by modus ponens)

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How do we leverage localness?

“Local” Equivalence for **Circuits**

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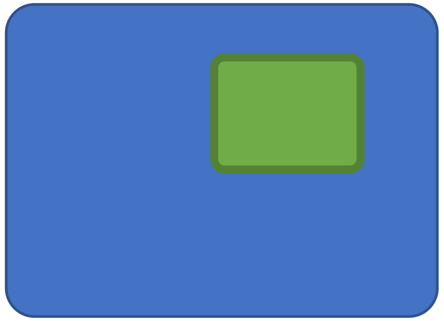
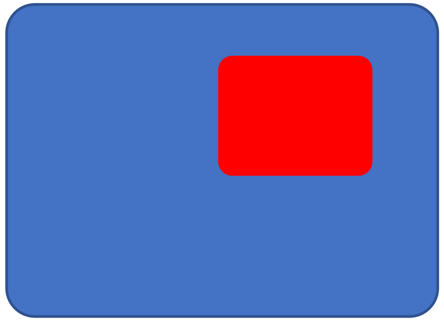
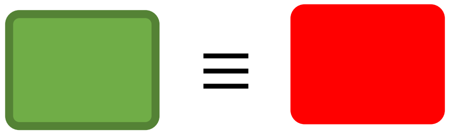
C: 

 : C'

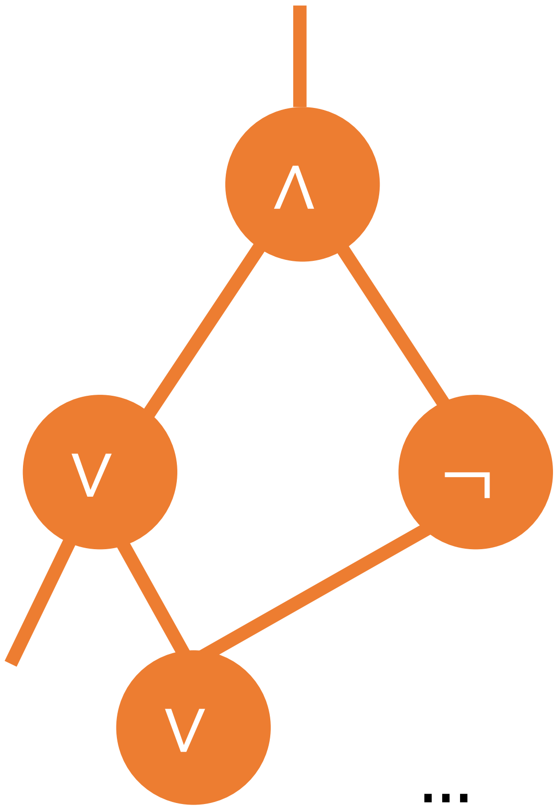
“Local” Equivalence for **Circuits**

C and C' are δ-equivalent, if C and $C'$are almost the same,

except for a **functionality equivalent *<u>sub-circuit</u>*** of size $O(\log n)$

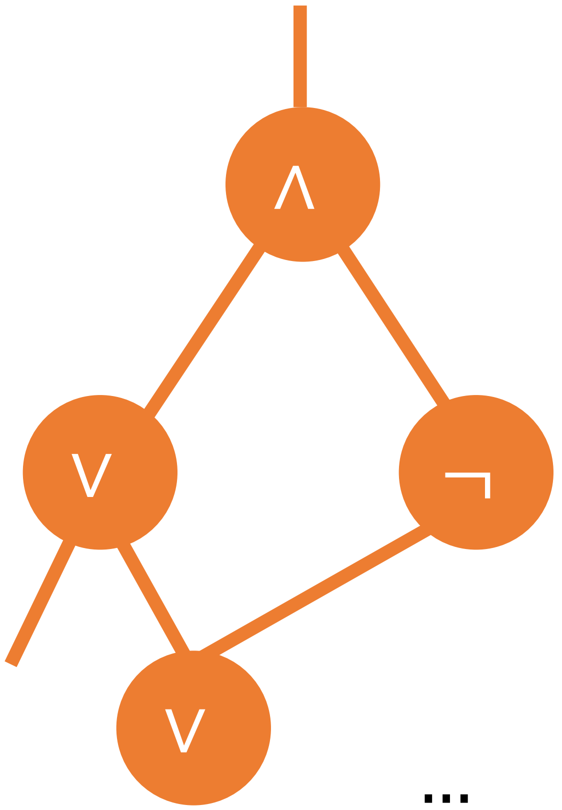


Sub-Circuits: Defined as Subsets of Gates



…

Sub-Circuits: Defined as Subsets of Gates



Directed Acyclic Graph

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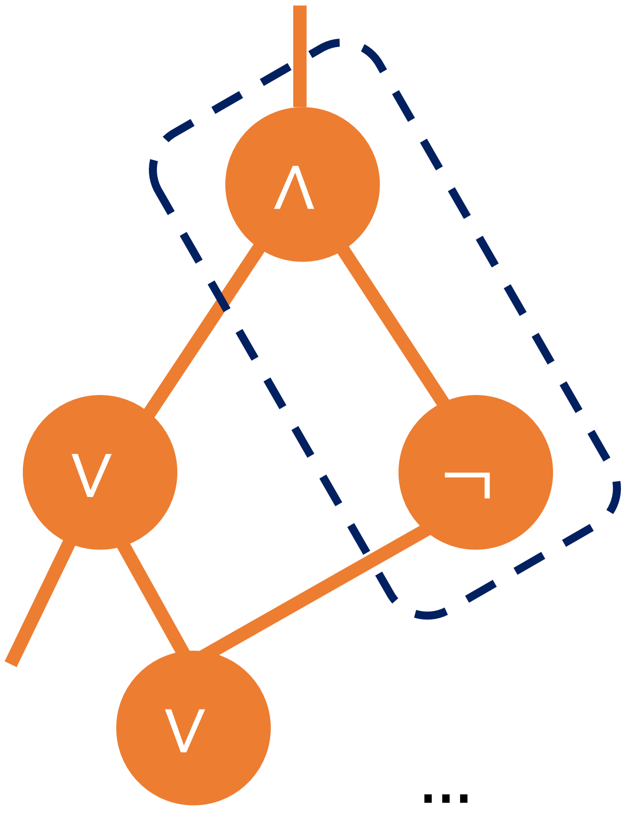

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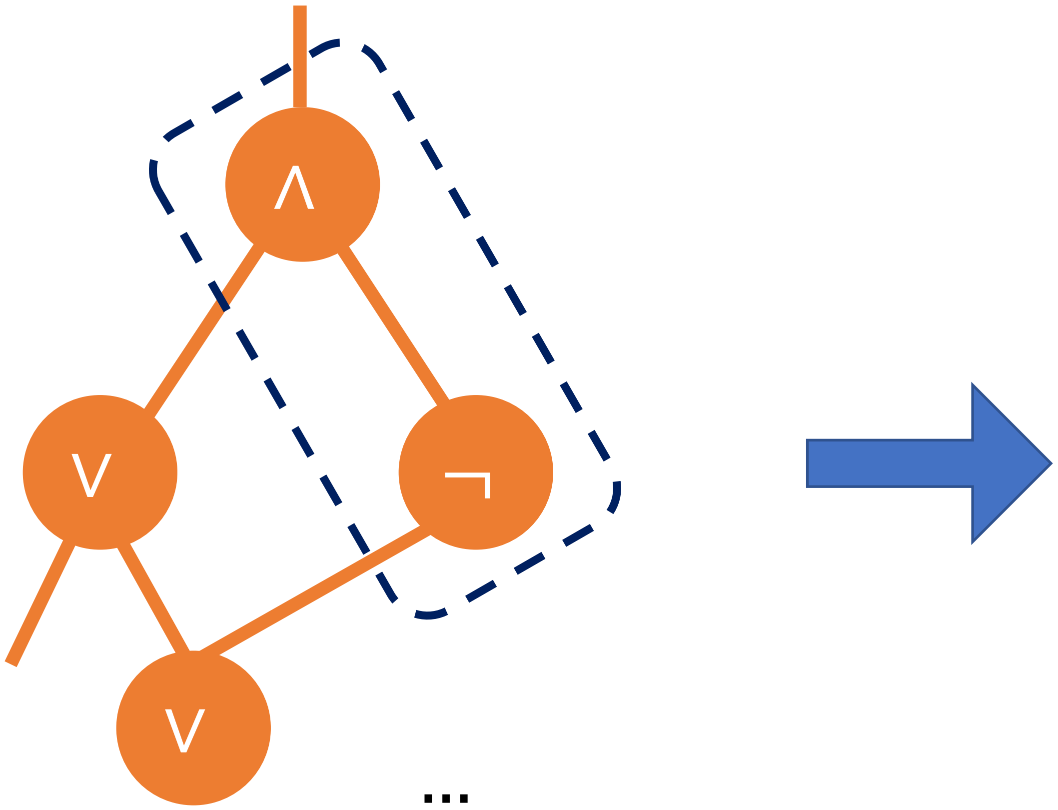

Directed Acyclic Graph

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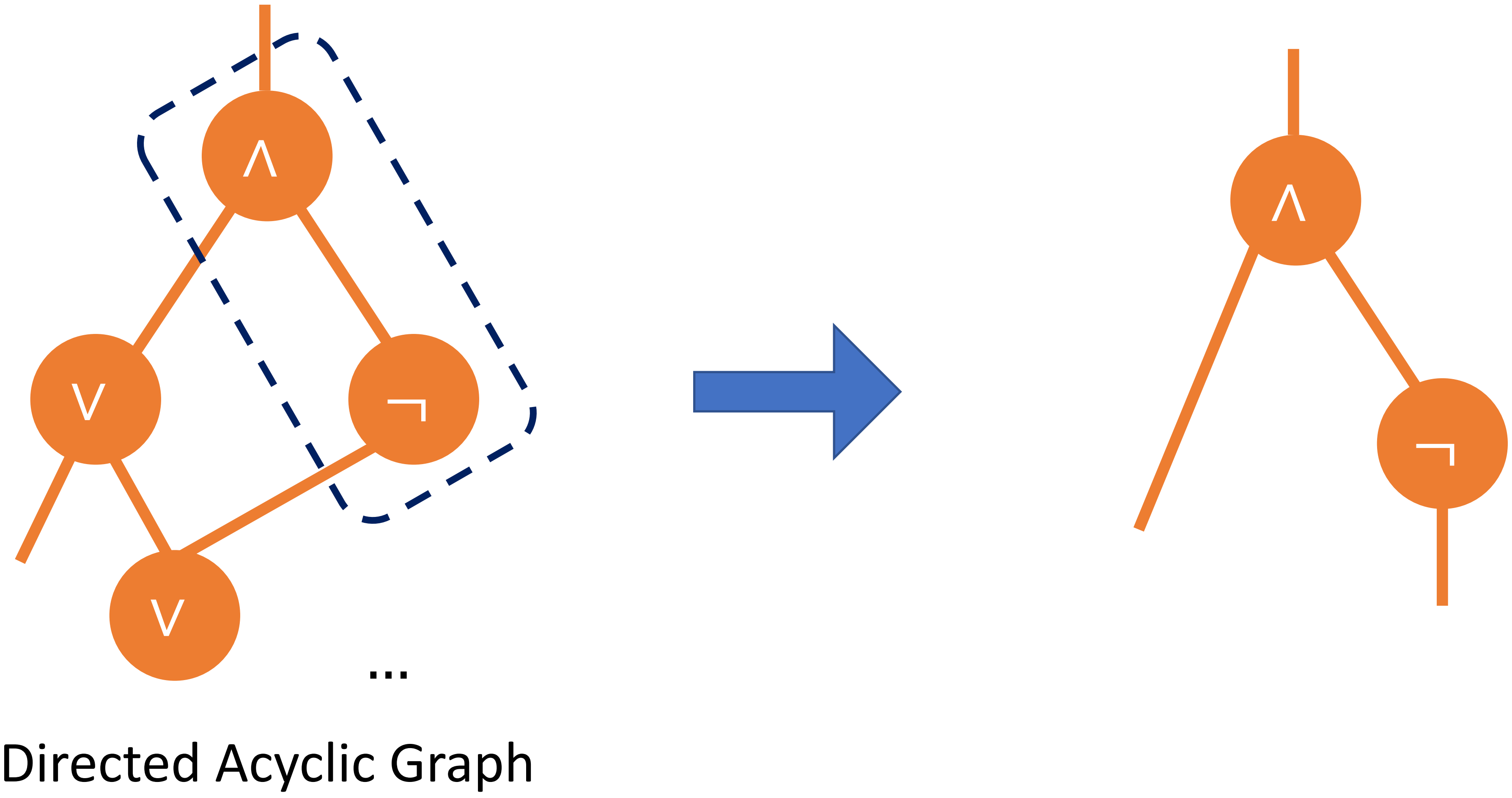

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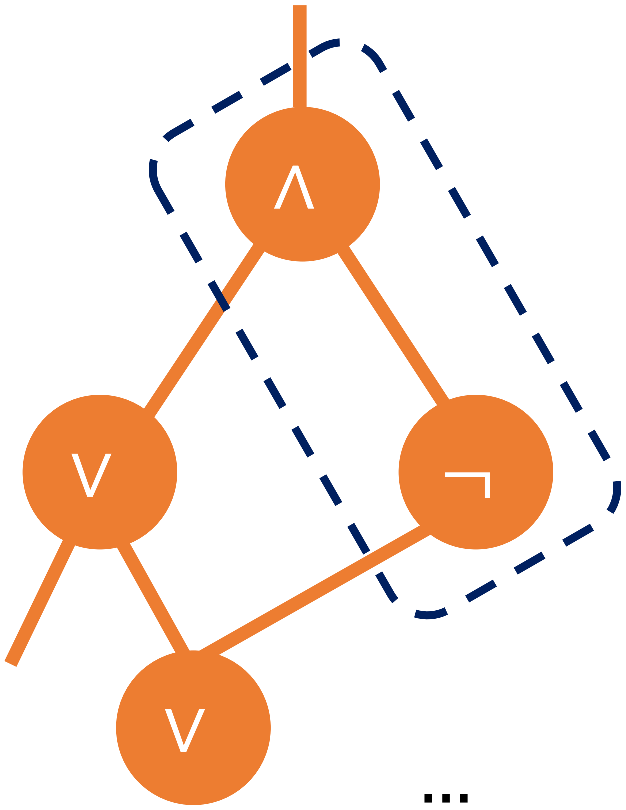
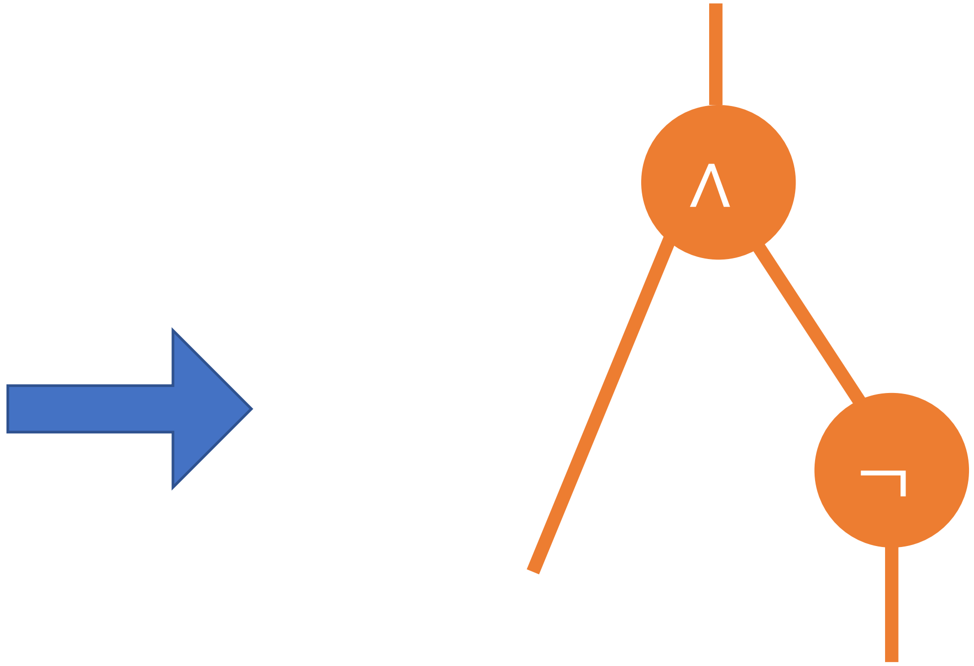
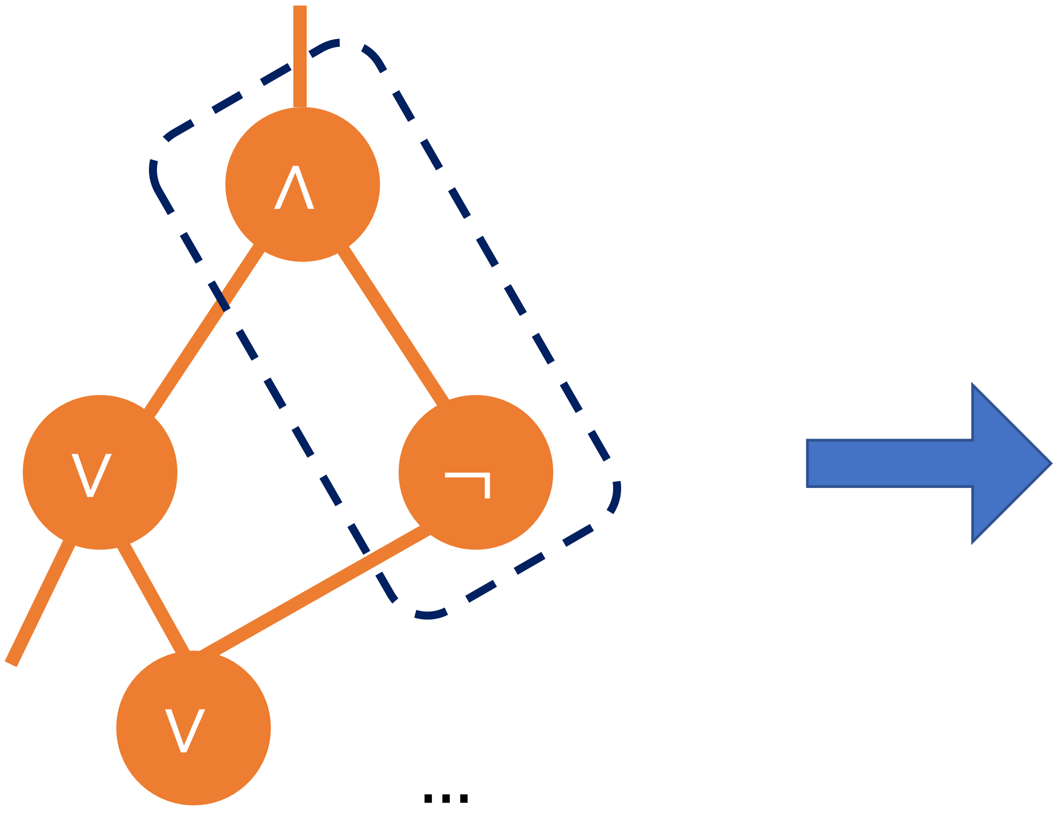


Directed Acyclic Graph

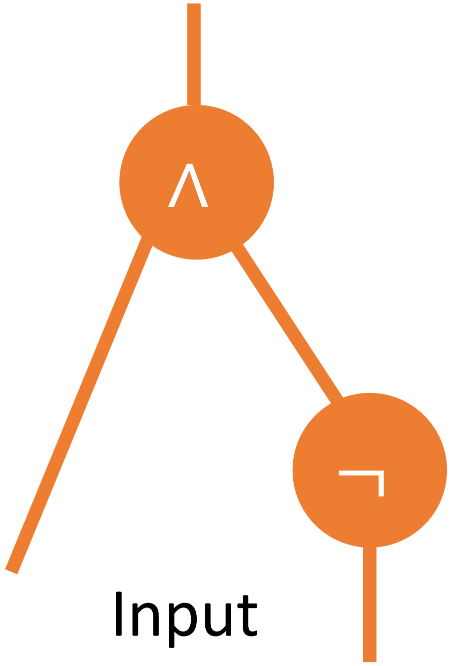


<u>**Sub-Circuit**</u>

Sub-Circuits: Defined as Subsets of Gates



Directed Acyclic Graph



Sub-Circuit

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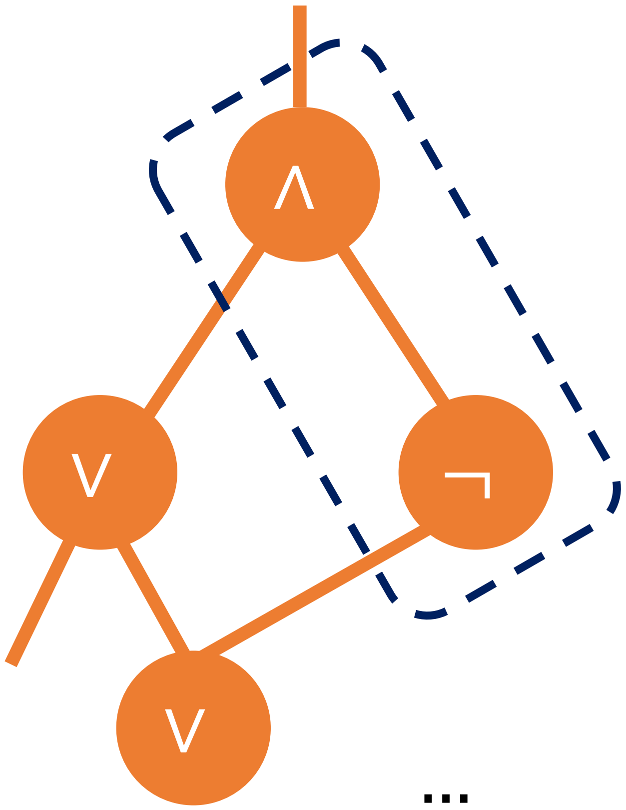
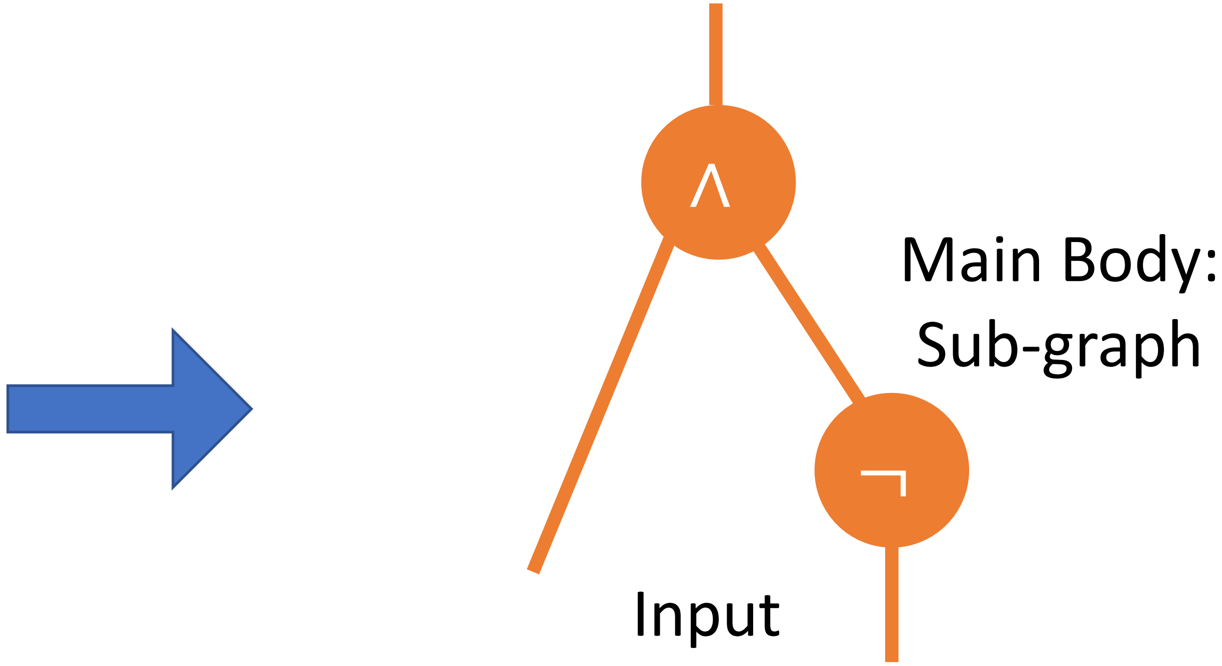
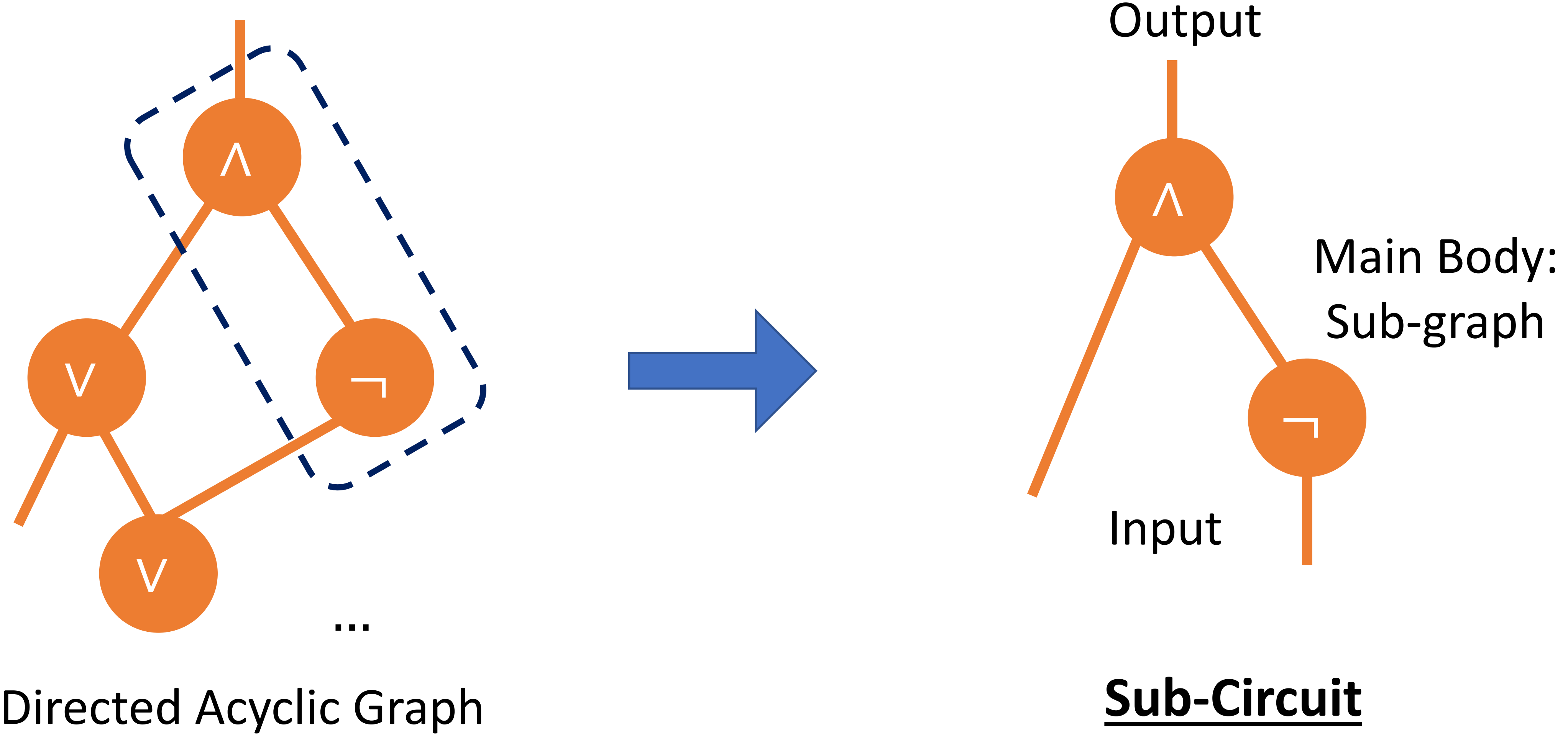


Directed Acyclic Graph

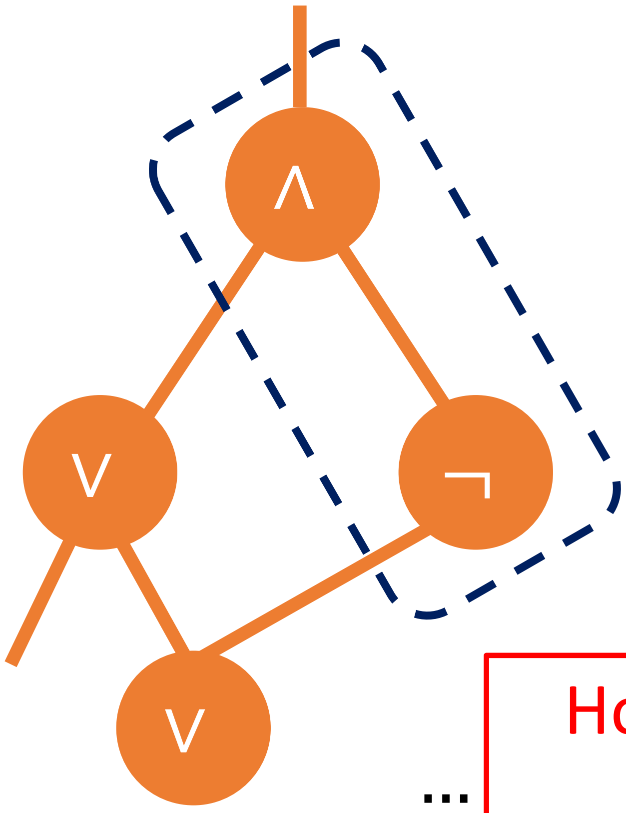


<u>Sub-Circuit</u>

Sub-Circuits: Defined as Subsets of Gates

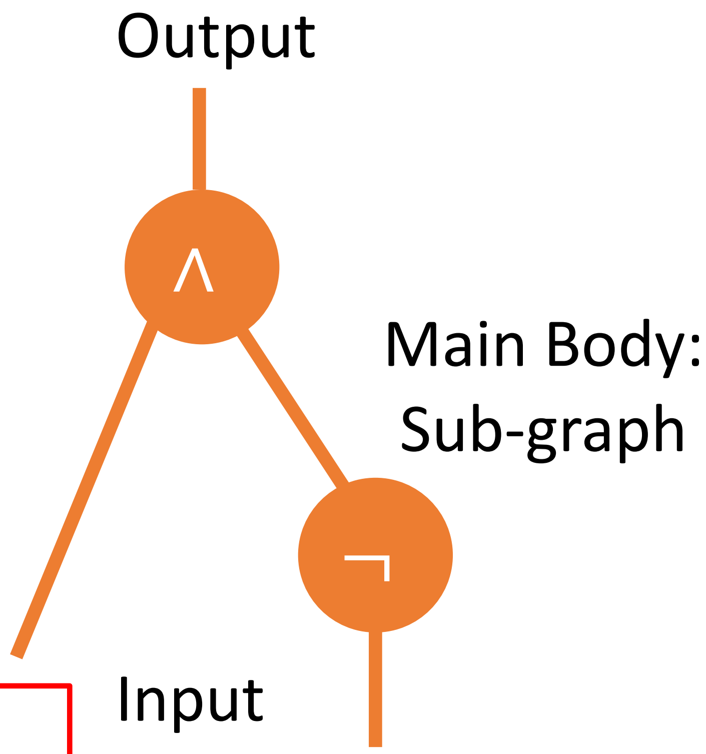


Sub-Circuits: Defined as Subsets of Gates



Directed Acyclic Graph

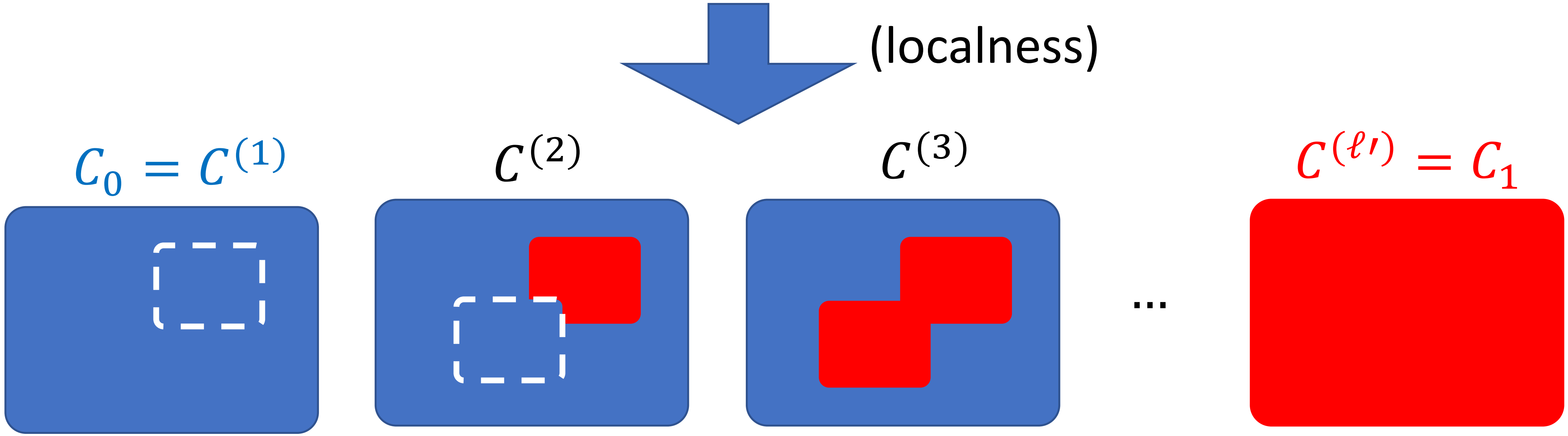
How does δ-Equivalence relate to proofs?



<u>Sub-Circuit</u>

$\mathcal{EF}$-Proofs imply δ-Equivalence

$\theta_1,\ \theta_2,\ \ldots,\ \theta_\ell$: $\mathcal{EF}$-proof of $C_0(x) \leftrightarrow C_1(x)$



$C^{(i)}$ and $C^{(i+1)}$ are δ-equivalent

Focus on δ-Equivalent Ckts

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Assume iO for δ-Equivalent Ckts exists: δiO

Focus on δ-Equivalent Ckts

Assume iO for δ-Equivalent Ckts exists: δiO

$\delta iO(C^{(1)})$ $\delta iO(C^{(2)})$ $\delta iO(C^{(3)})$ … $\delta iO(C^{(\ell')})$

Focus on δ-Equivalent Ckts

Assume iO for δ-Equivalent Ckts exists: δiO

$$\delta iO(C^{(1)}) \quad \approx \quad \delta iO(C^{(2)}) \quad \approx \quad \delta iO(C^{(3)}) \quad \dots \quad \delta iO(C^{(\ell')})$$

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$$\Rightarrow \delta iO(C_0) \approx_c \delta iO(C_1)$$

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<u>Total Security Loss</u> = ℓ' ·Loss of δiO $\quad (\ell' = poly)$

Focus on δ-Equivalent Ckts

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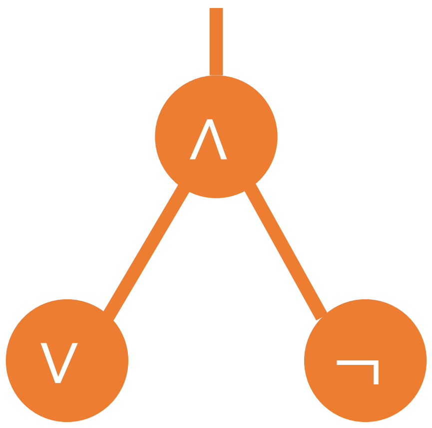
Total Security Loss = ℓ' $\cdot$Loss of δiO $\qquad (\ell' = poly)$

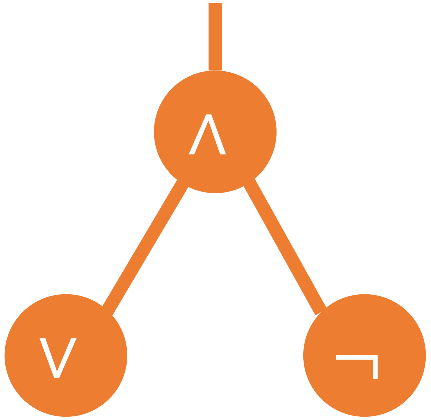
If loss of δiO is independent of $|input|$, so is the total loss.

Constructing δiO

Constructing δiO



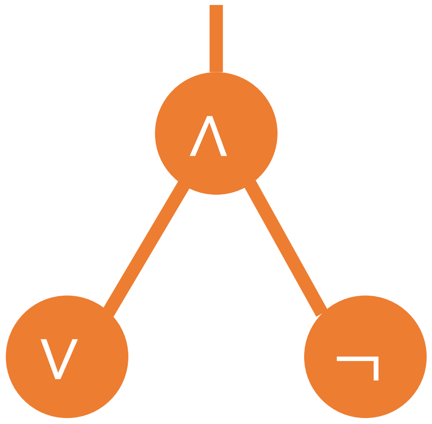
Constructing δiO



"Gate-by-Gate" Obfuscation:

Obfuscate each gate separately

Constructing δiO



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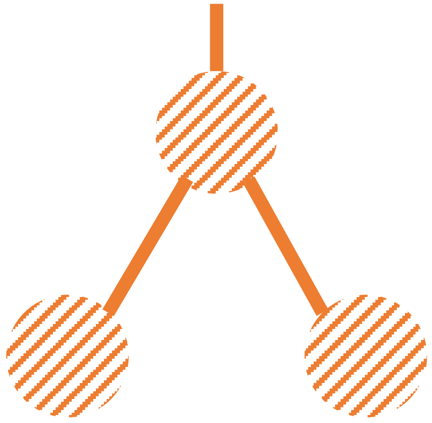

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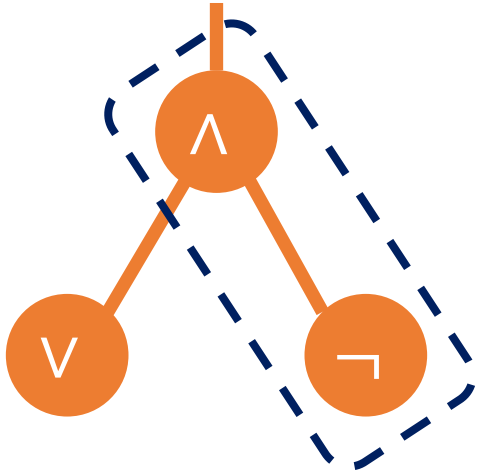

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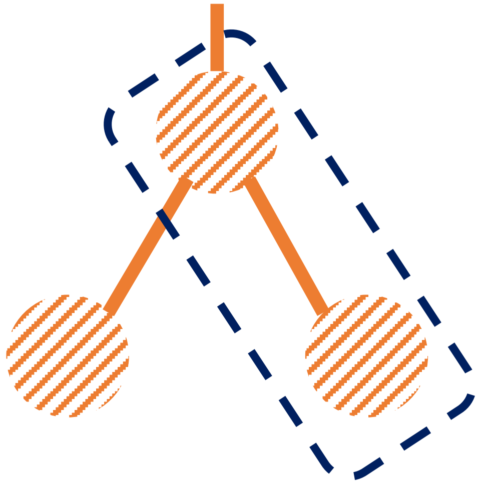

Security loss is independent of $|input|$, why?

We can "**cut**" the obfuscated program!

Security Proof for δiO

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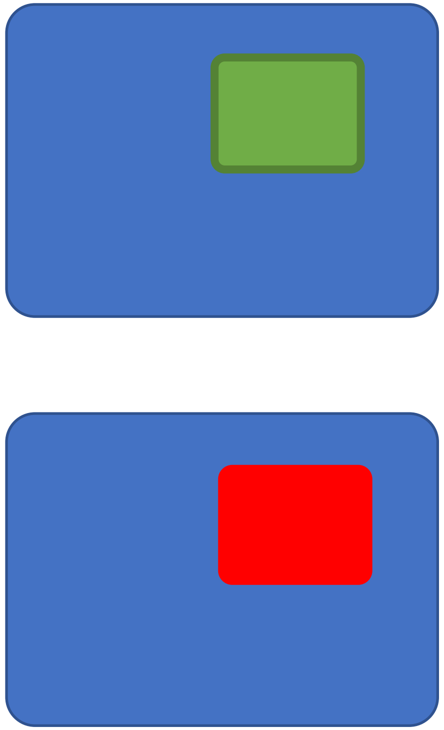
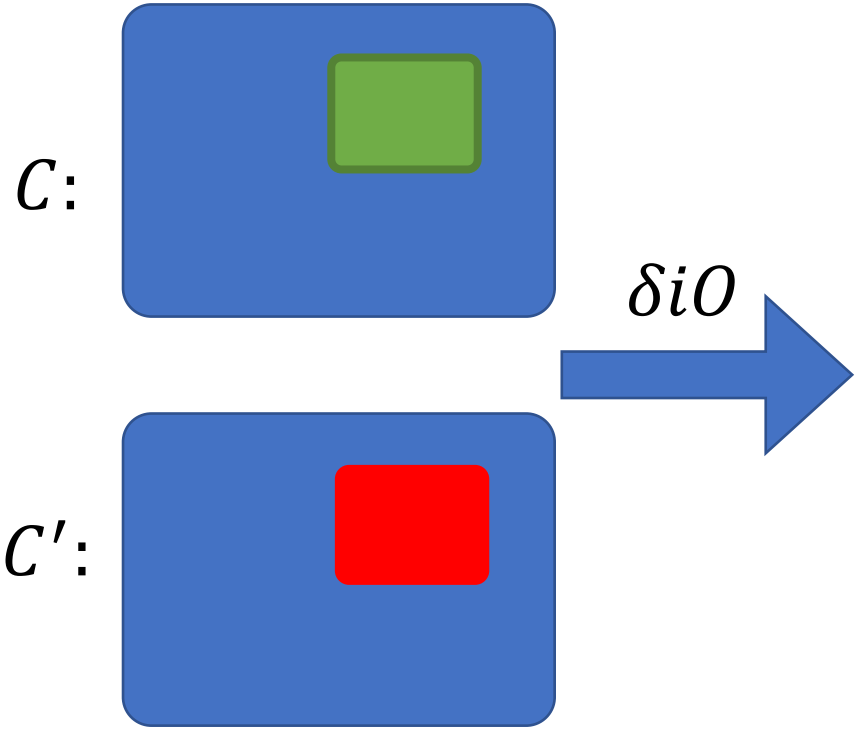
C:

C':



C, C': δ-Equivalent

Security Proof for δiO



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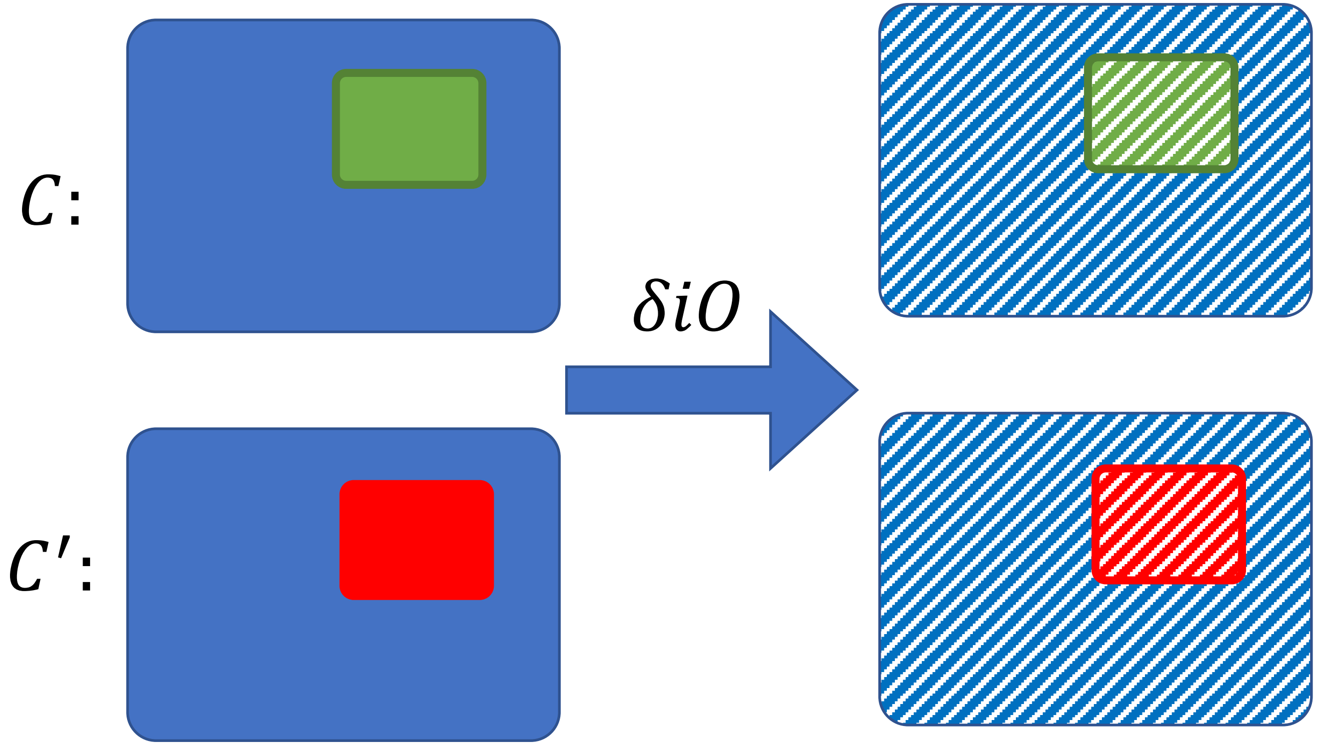

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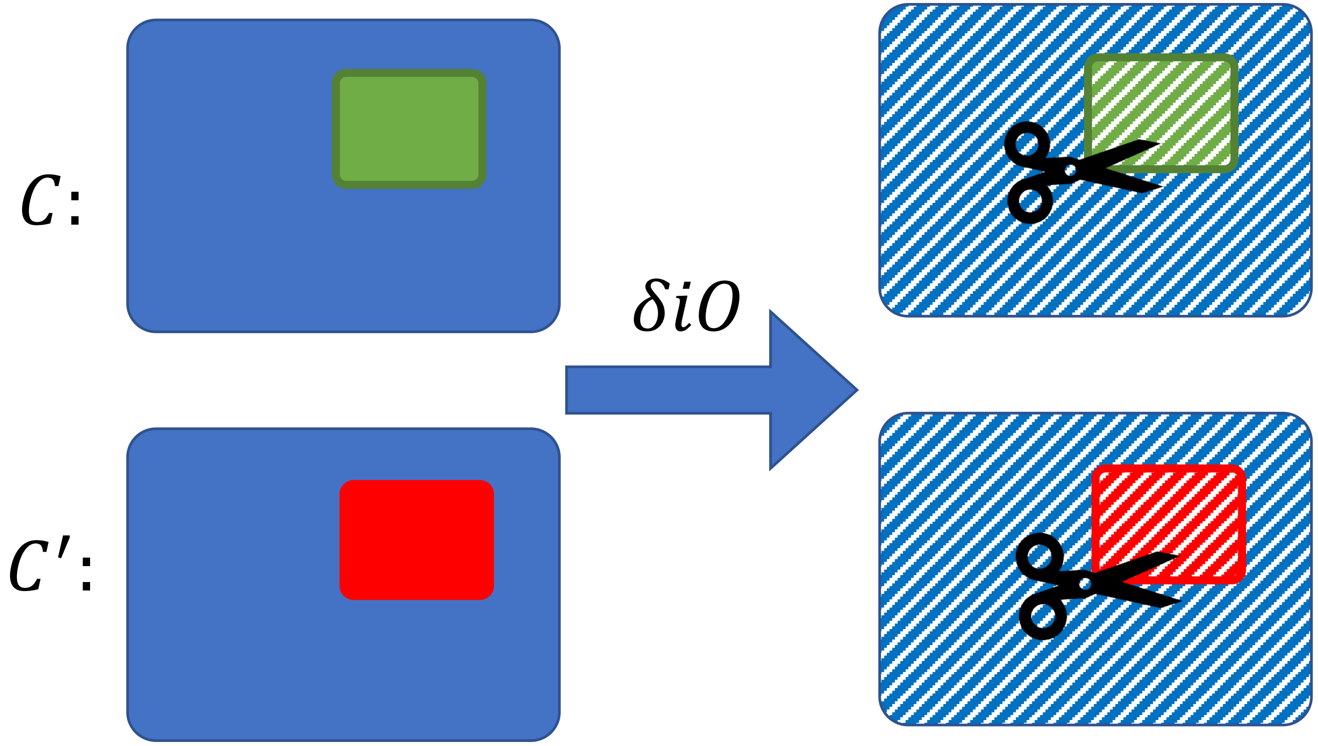

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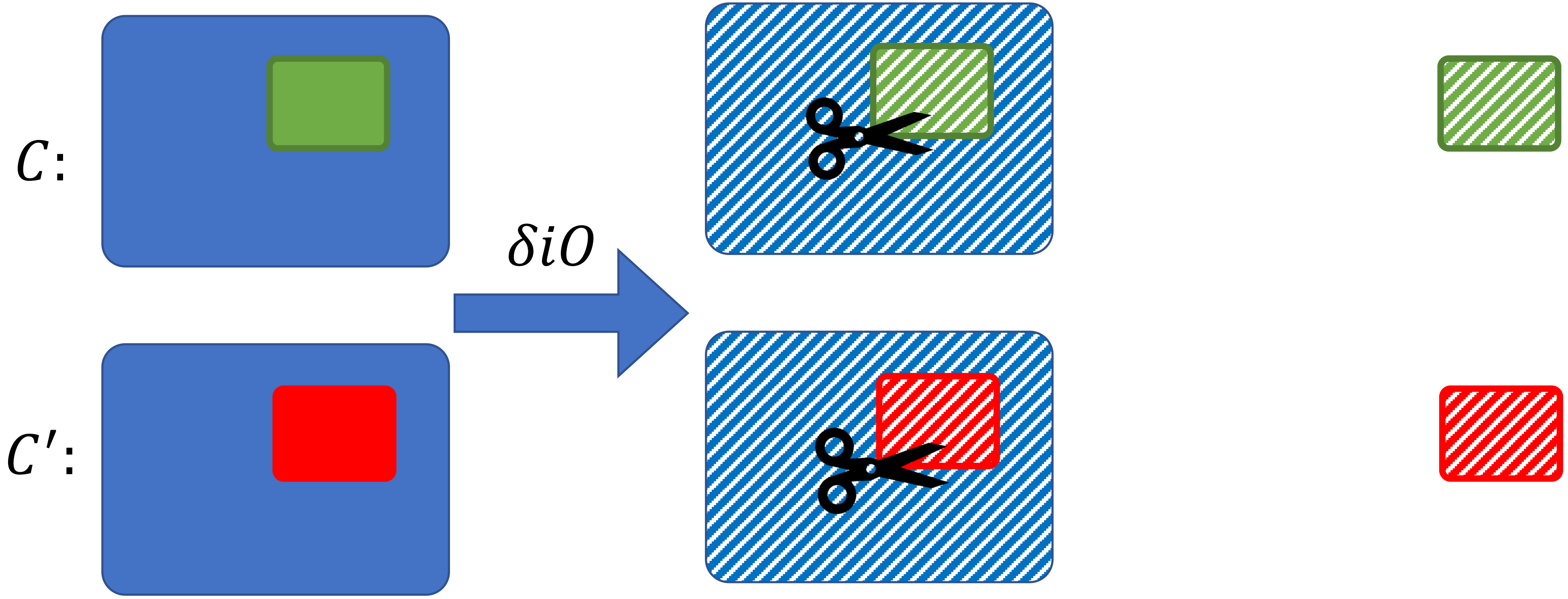

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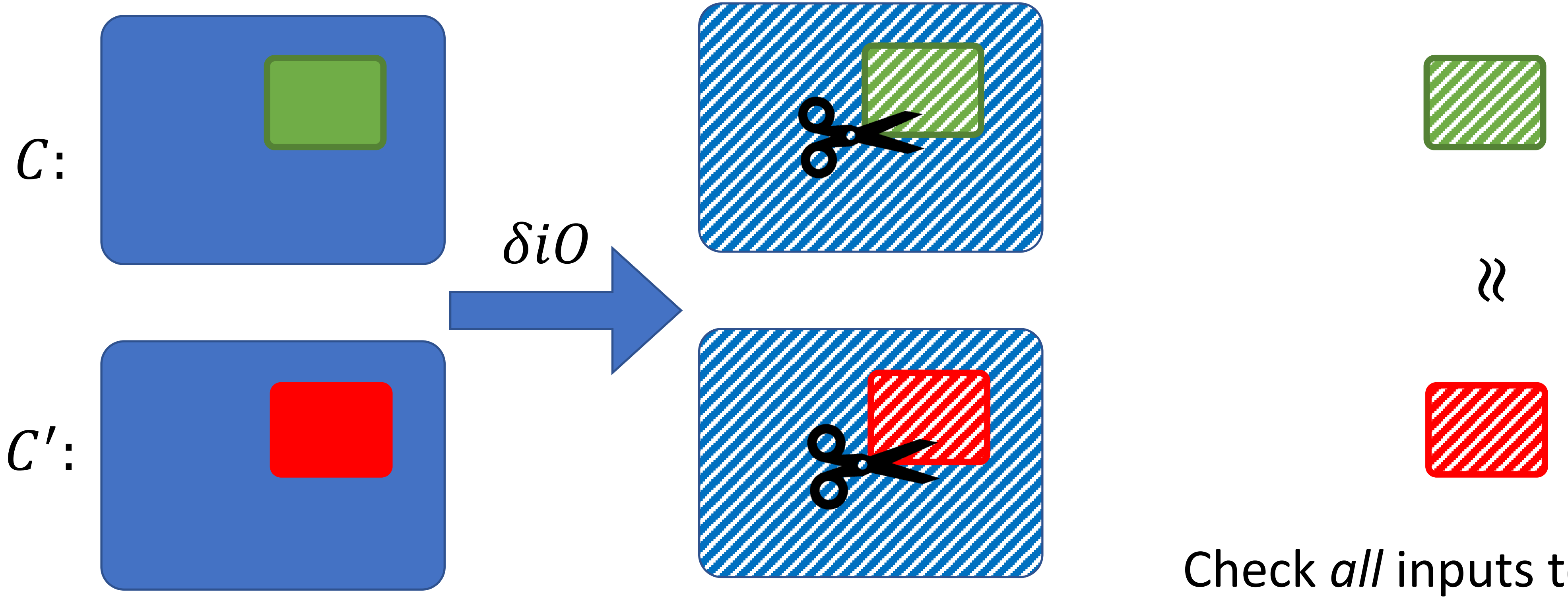

C, C': δ-Equivalent

Check *all* inputs to **Sub-ckt**

Security Loss: $2^{|subckt\ input|}$

$= 2^{O(\log n)} = poly$!

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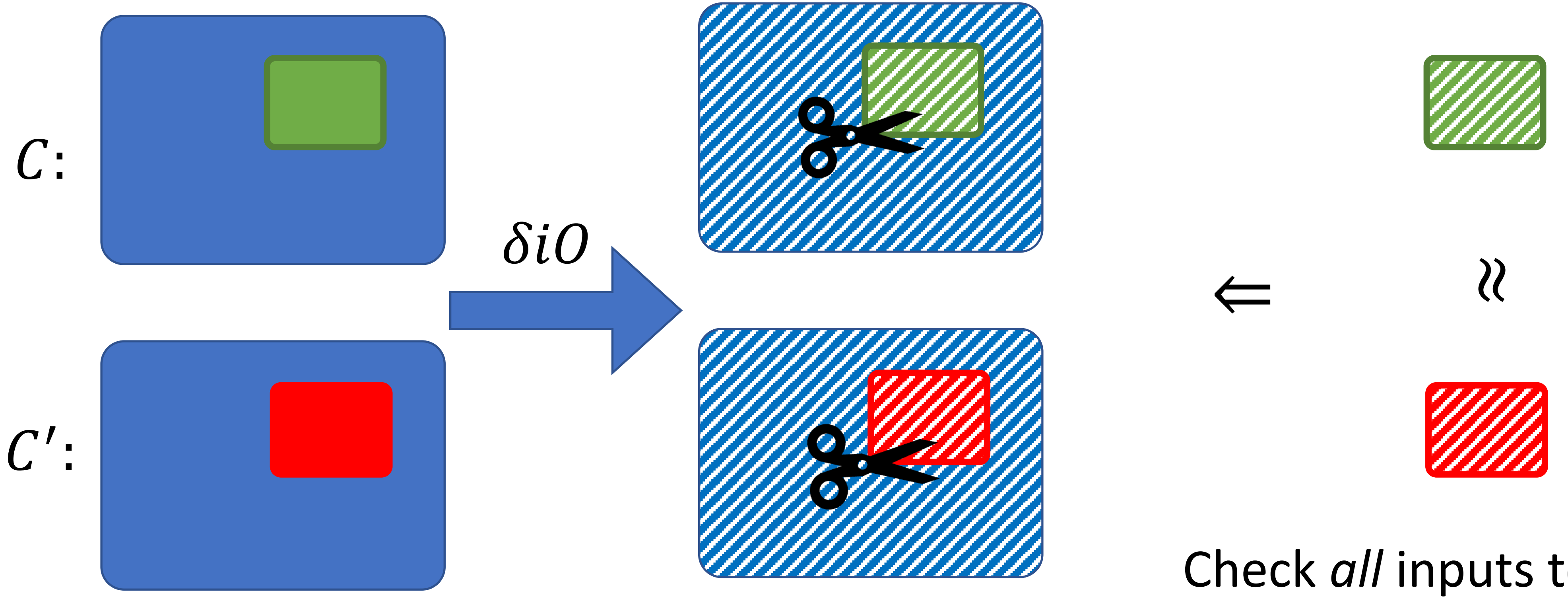

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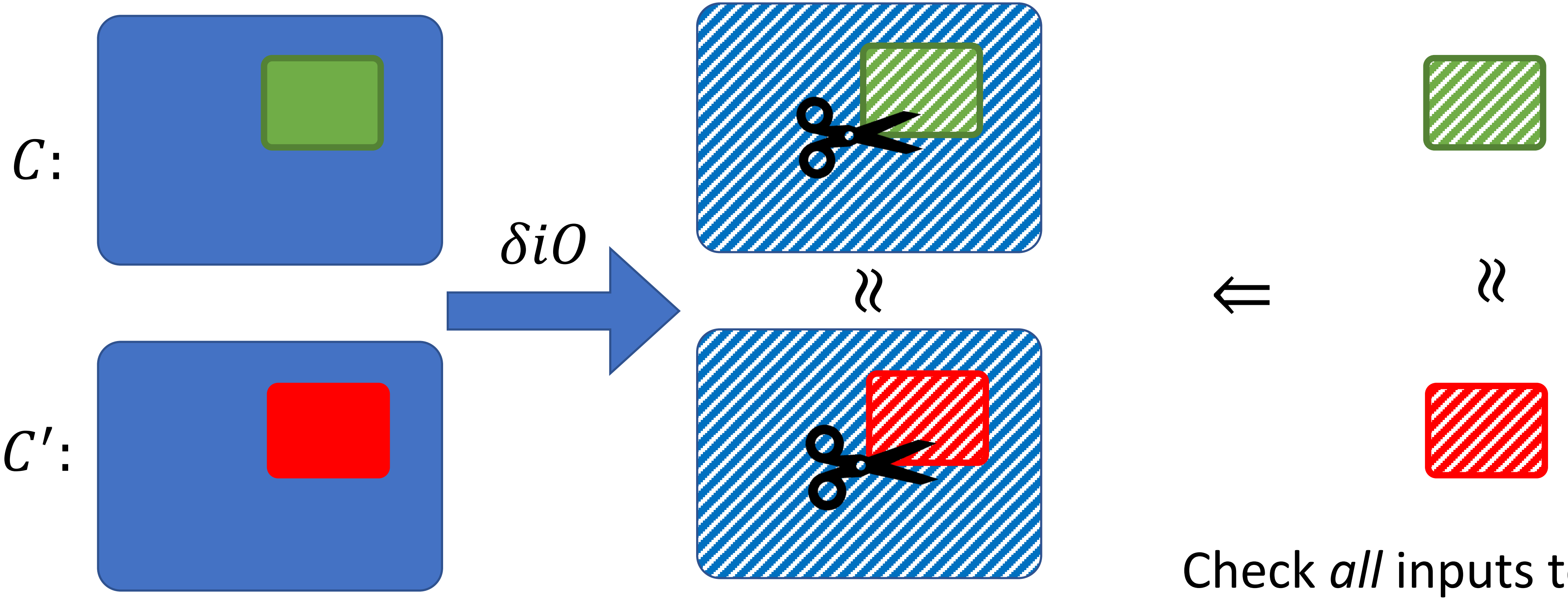

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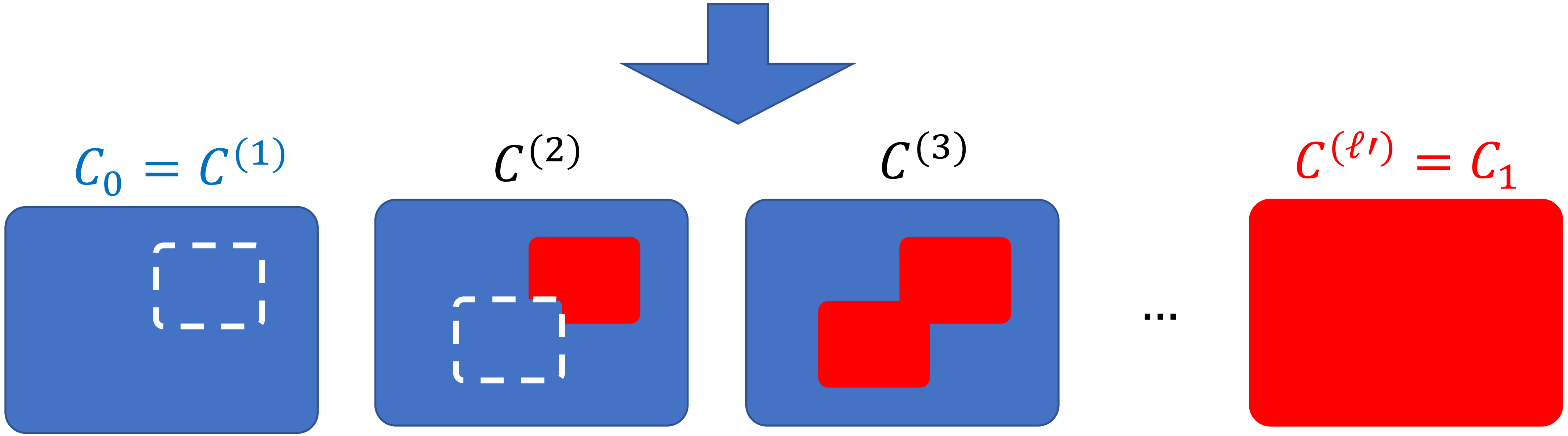
Technical Details

- **$\boldsymbol{\delta}$-Equivalence from $\mathcal{EF}$-proofs**
- δiO Construction
- iO for Turing Machines

Recall: δ-Equivalence via $\mathcal{EF}$-Proofs

$\theta_1,\ \theta_2,\ \ldots,\ \theta_\ell$: $\mathcal{EF}$-proof of $C_0(x) \leftrightarrow C_1(x)$



Recall: δ-Equivalence via $\mathcal{EF}$-Proofs

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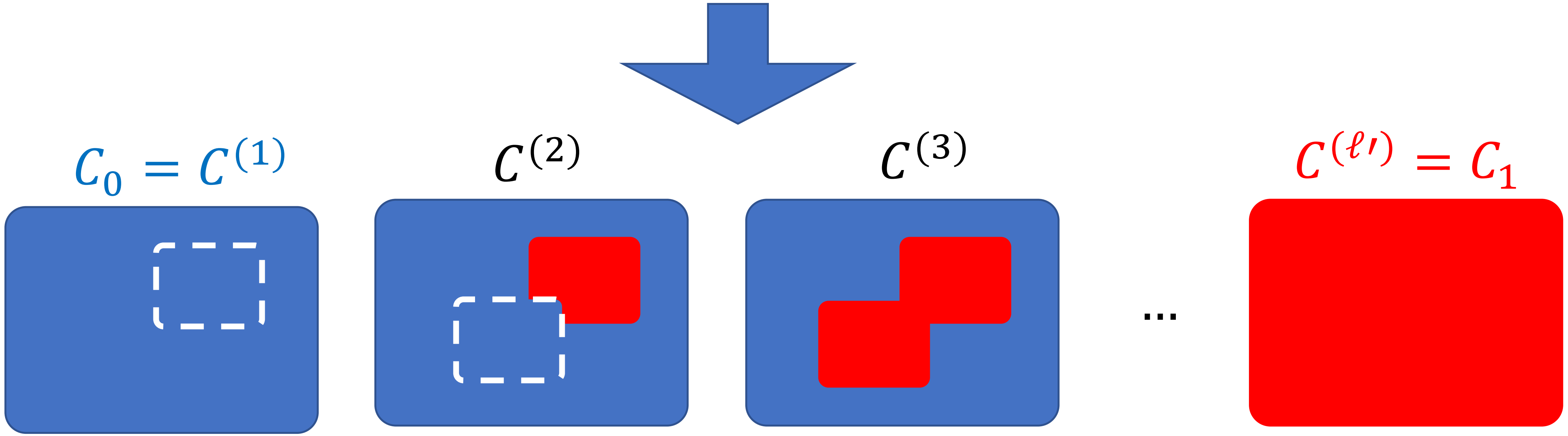

A sequence of **incremental** changes from C_0 to C_1

$C_0 \to C_1$, Stage I: “Grow” C_1

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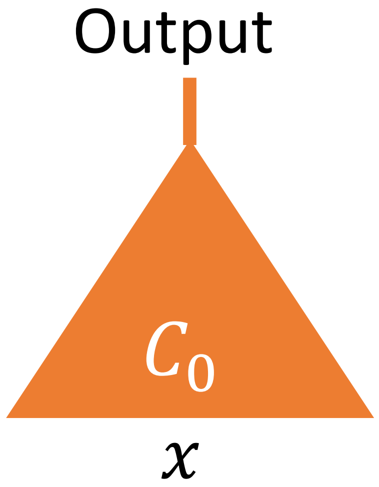

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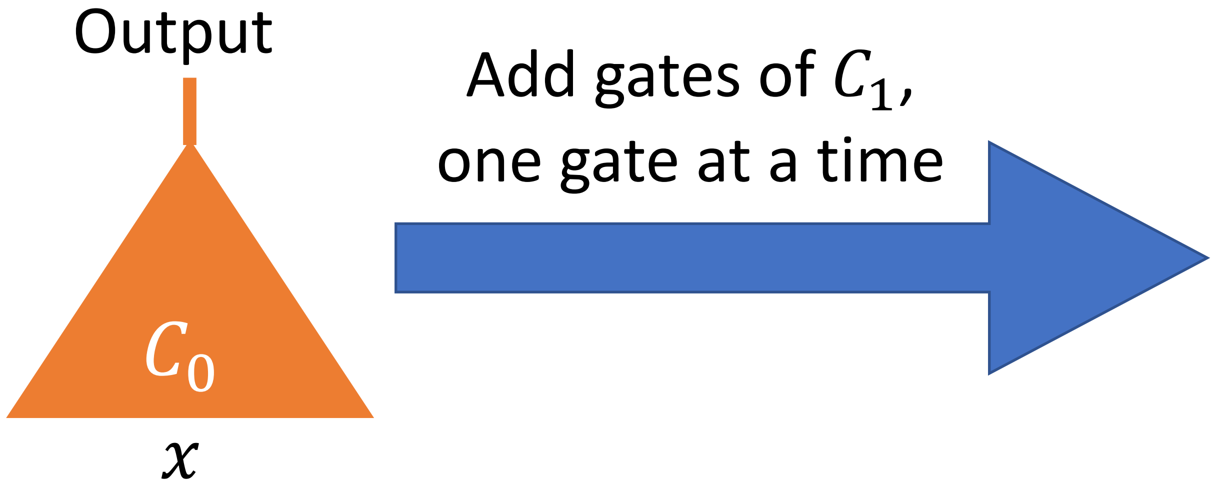

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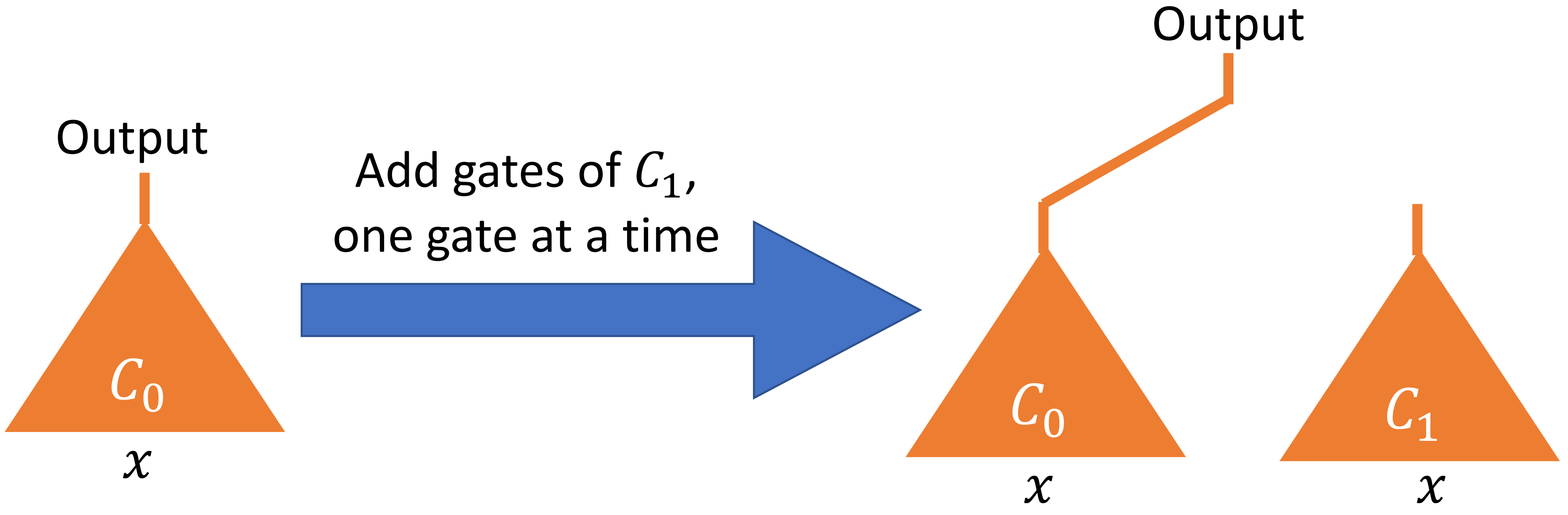

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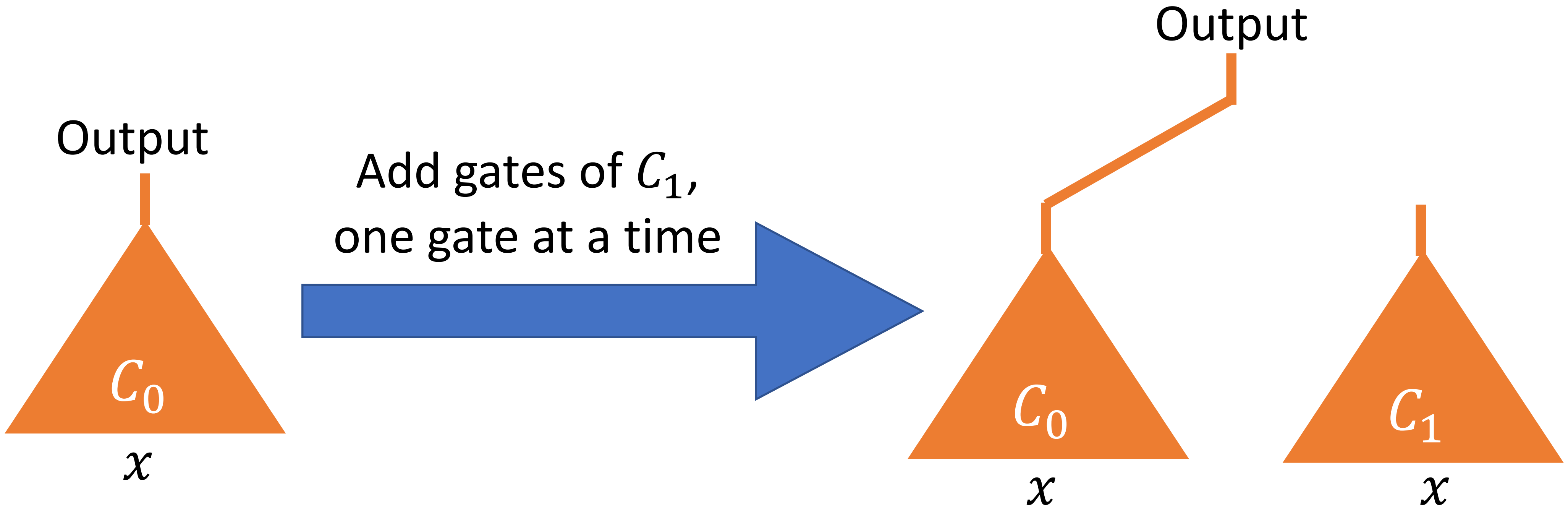
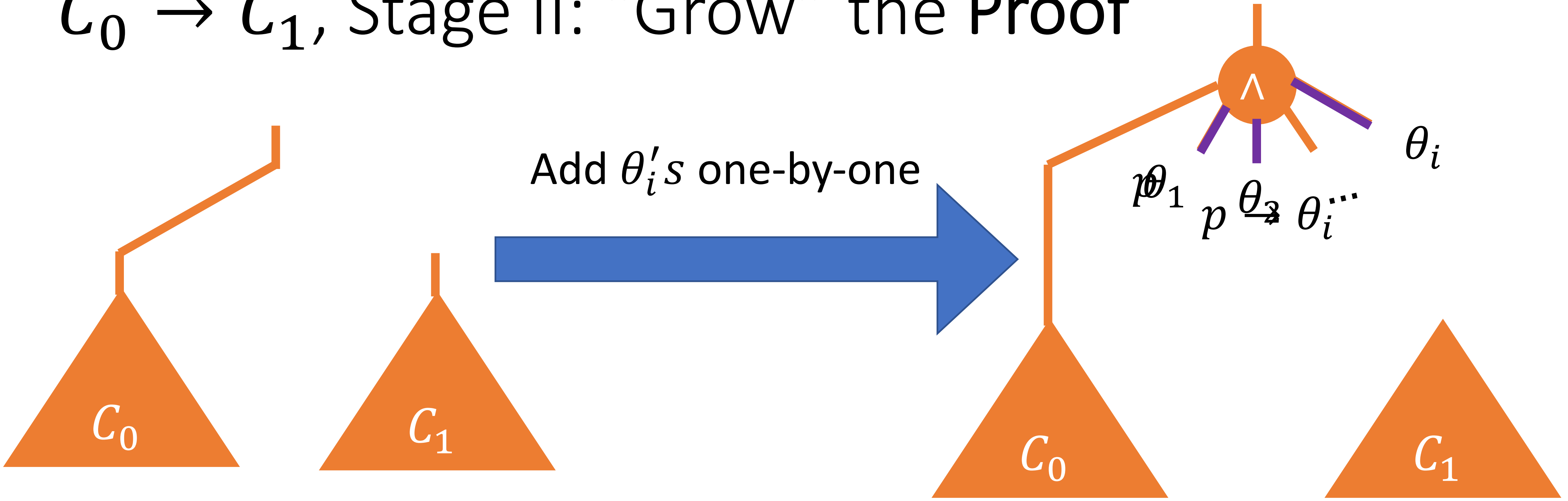


δ-Equivalence:

- We only add 1 gate at a time
- Gate we add doesn't affect output

$C_0 \to C_1$, Stage II: "Grow" the **Proof**

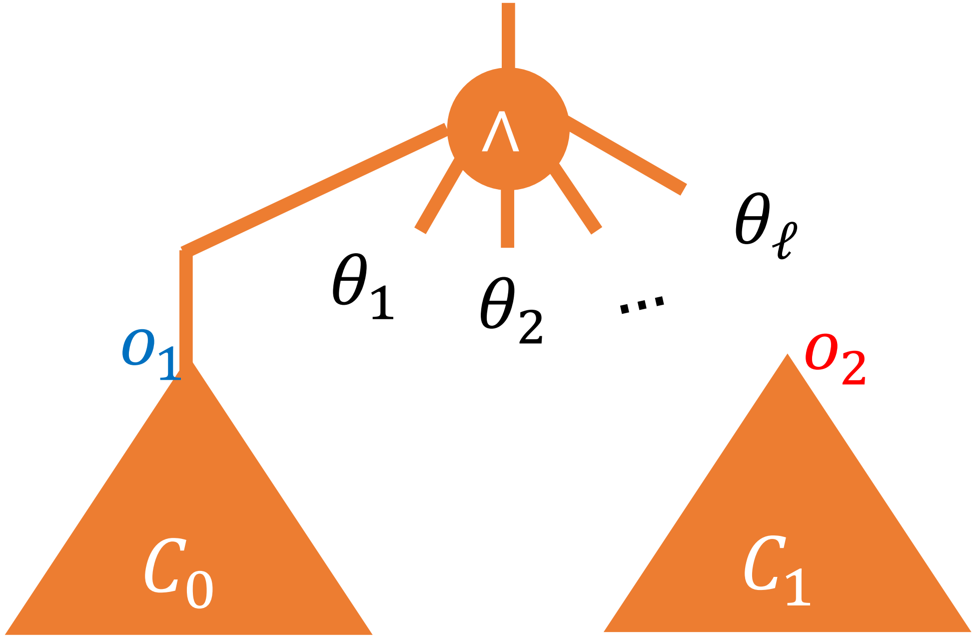


***δ*-Equivalence**: θ_i is from...

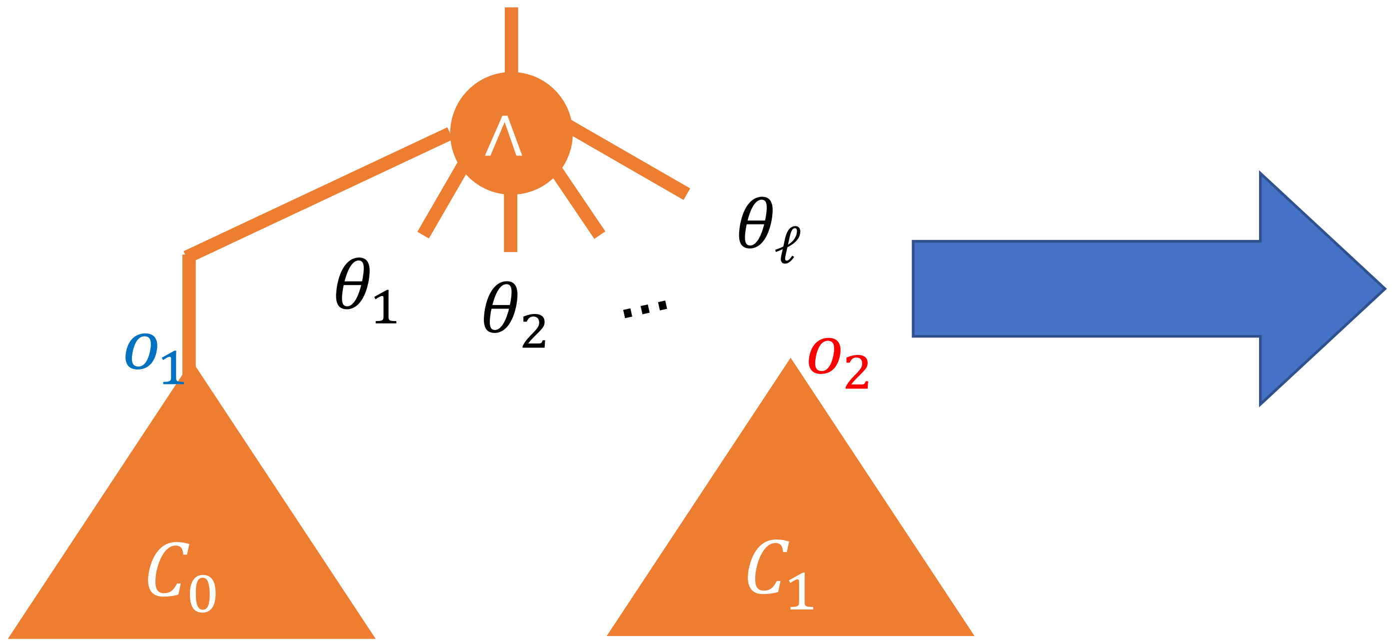
- Axiom: θ_i a tautology, itself is the sub-ckt
- Modus Ponens: $p \wedge (p \to \theta_i) = p \wedge (p \to \theta_i) \wedge \theta_i$

$C_0 \to C_1$, Stage III: **Switch** the Output

$C_0 \rightarrow C_1$, Stage III: **Switch** the Output



$C_0 \to C_1$, Stage III: **Switch** the Output



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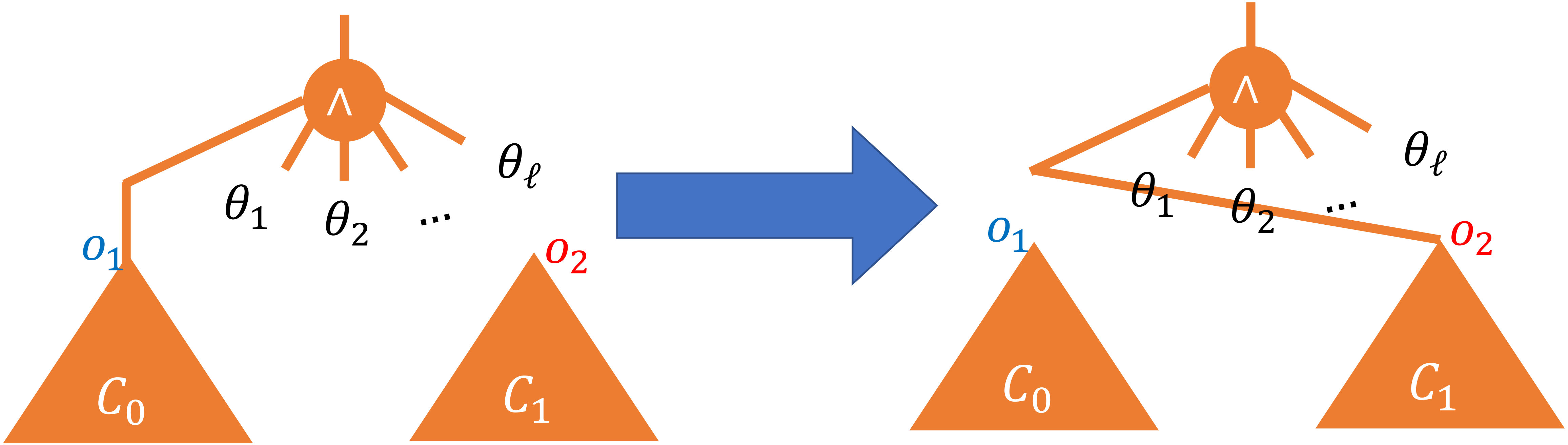

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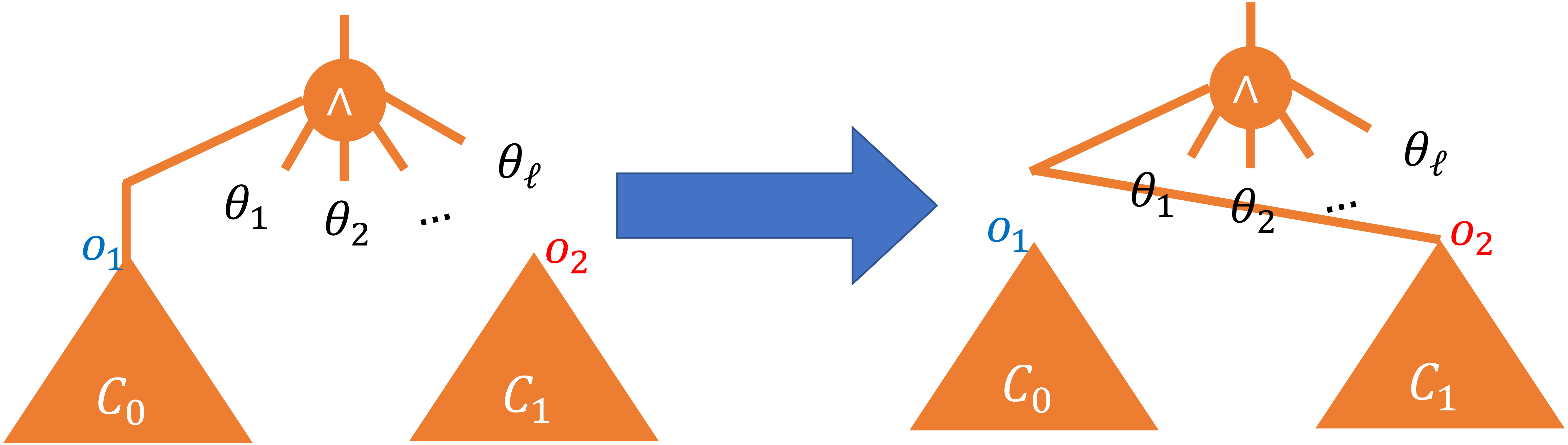

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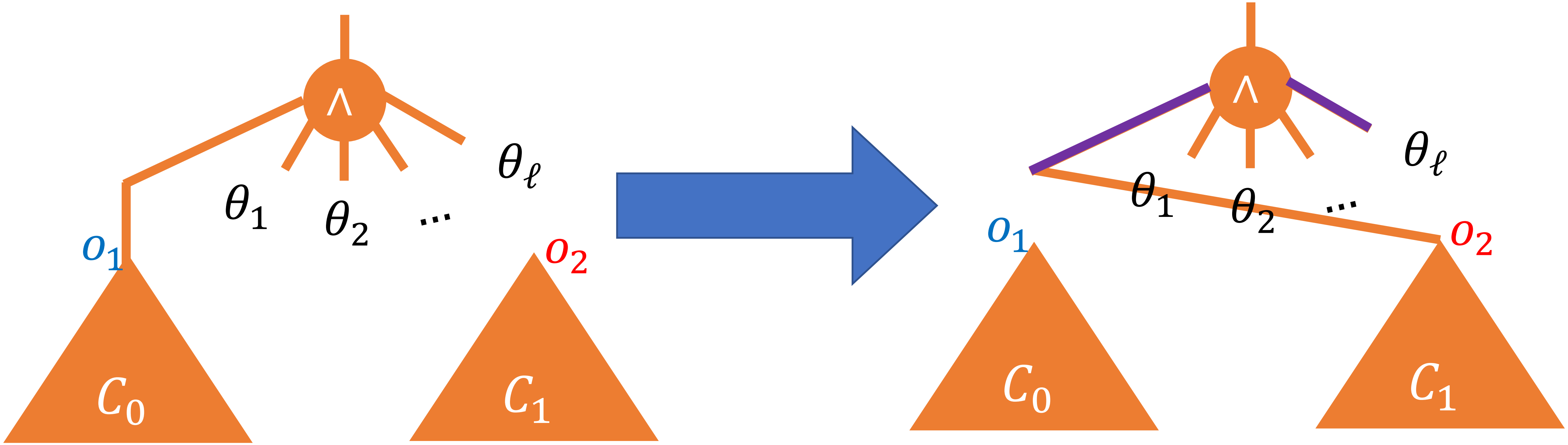

***δ*-Equivalence**: $\theta_\ell =$ "$o_1 \leftrightarrow o_2$",

$$o_1 \wedge (o_1 \leftrightarrow o_2) = o_2 \wedge (o_1 \leftrightarrow o_2)$$

$C_0 \to C_1$, Stage IV: "**Shrink**" the Proof

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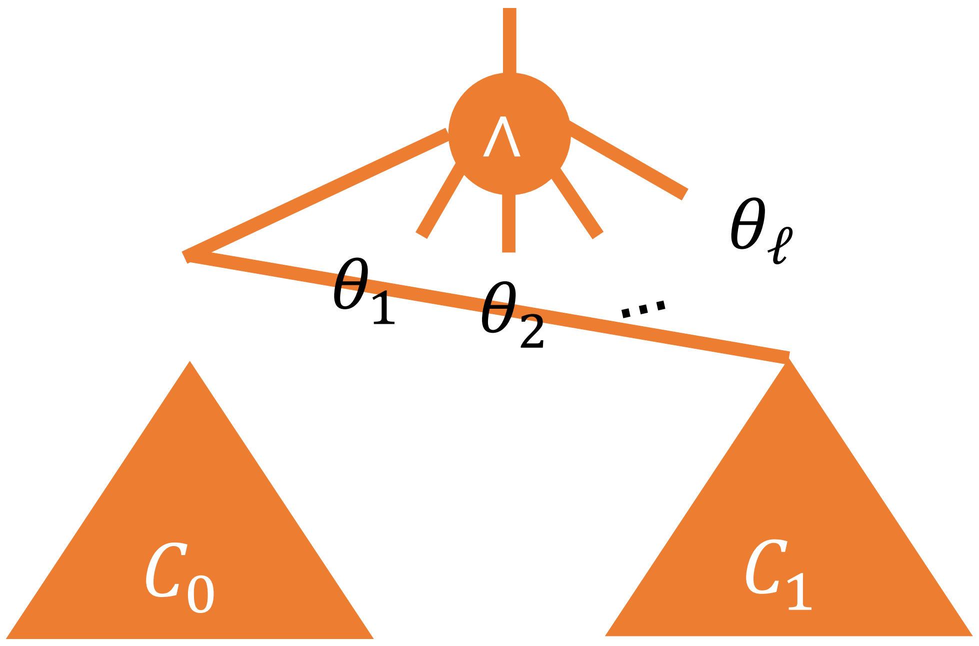
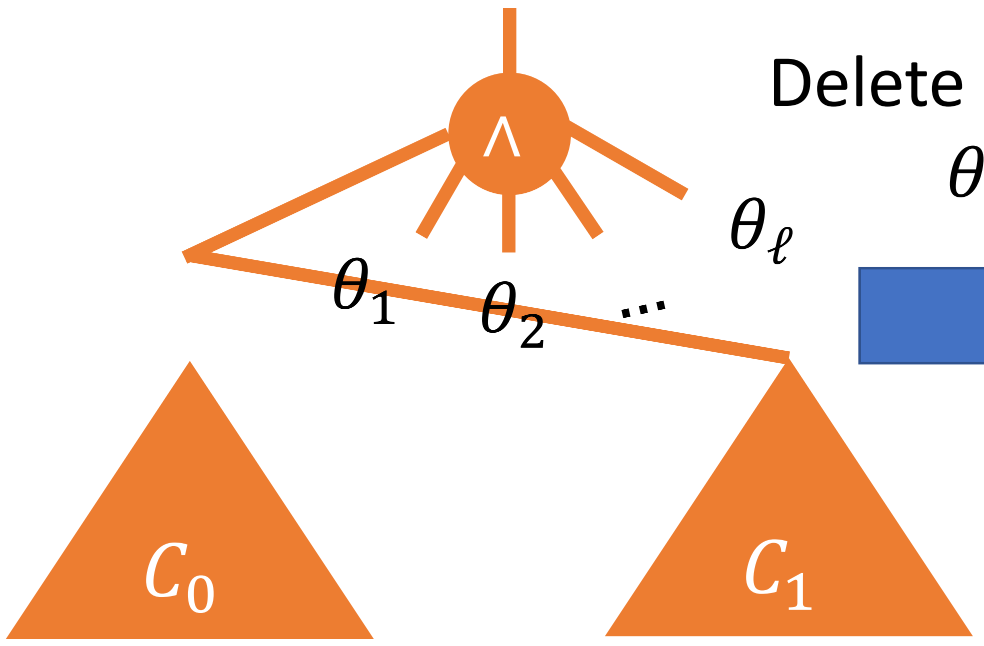
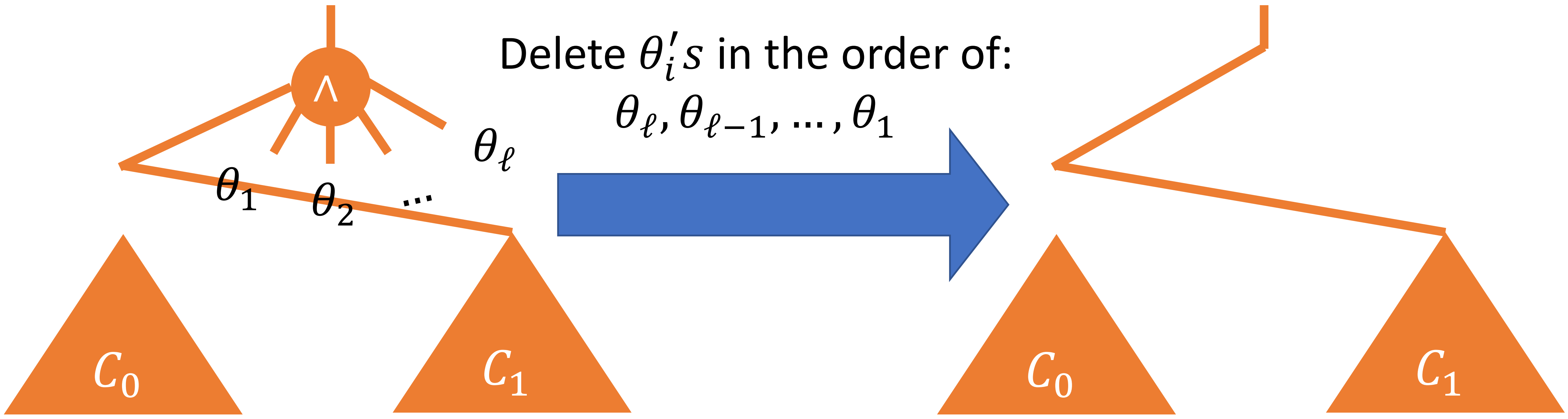


$C_0 \rightarrow C_1$, Stage IV: "**Shrink**" the Proof



Delete $\theta_i' s$ in the order of:
$\theta_\ell, \theta_{\ell-1}, \dots, \theta_1$

$C_0 \to C_1$, Stage IV: "**Shrink**" the Proof



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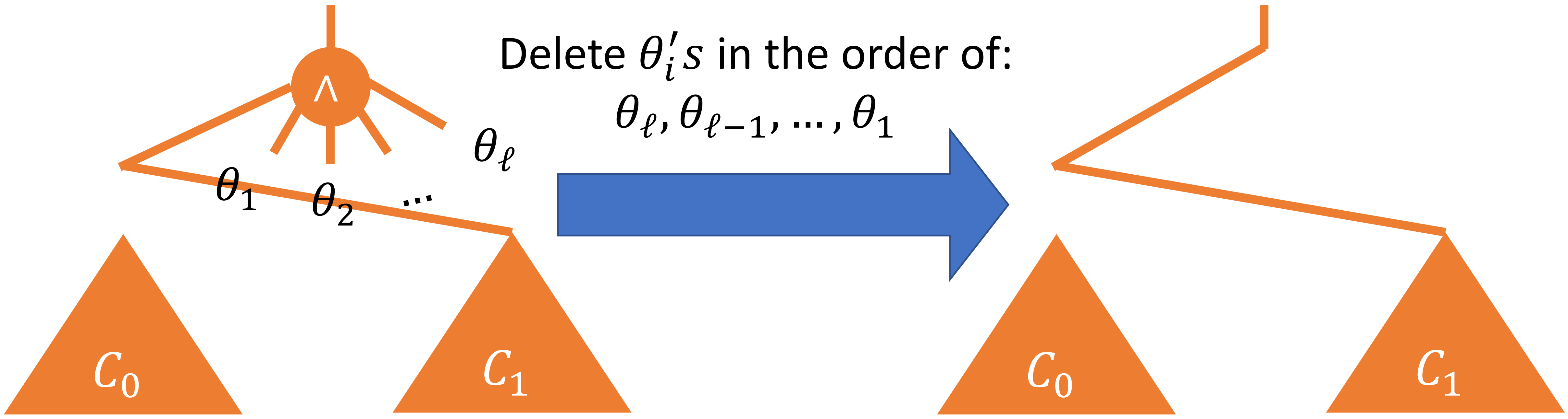

δ-Equivalence: same as the growing phase

$C_0 \to C_1$, Stage V: “Shrink” C_0

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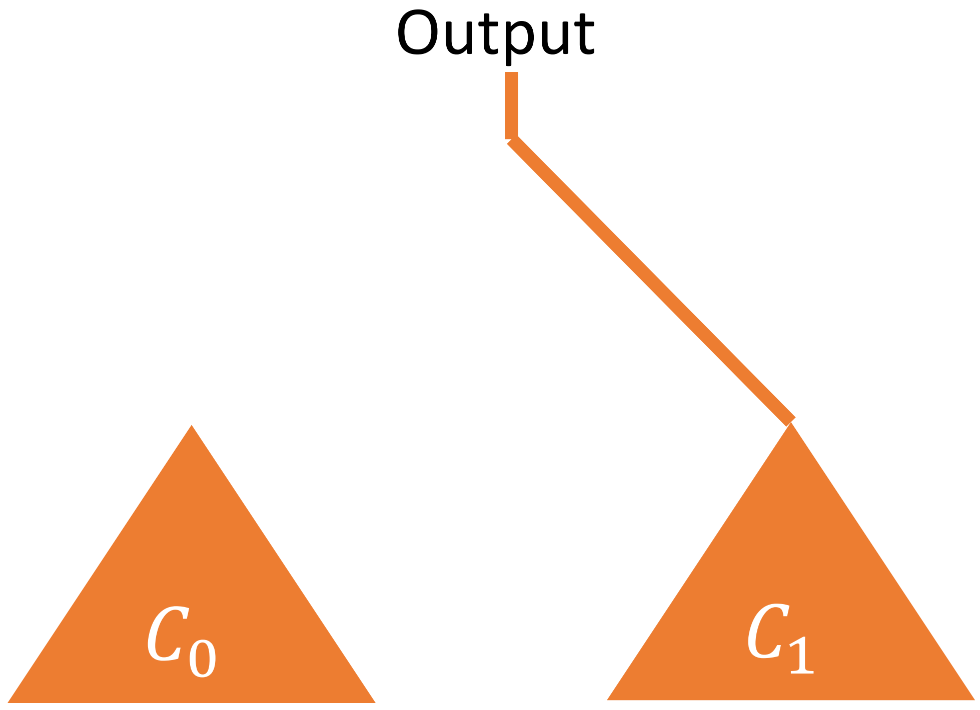
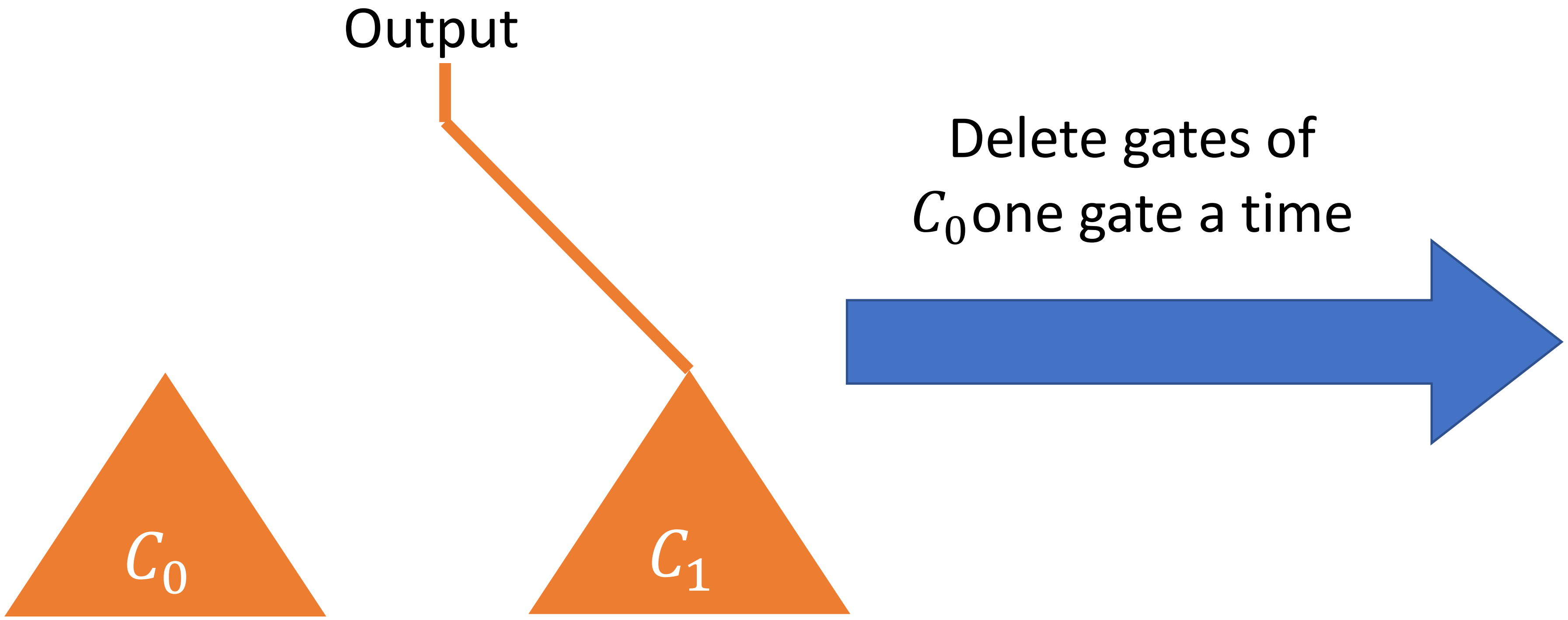


$C_0 \rightarrow C_1$, Stage V: "Shrink" C_0



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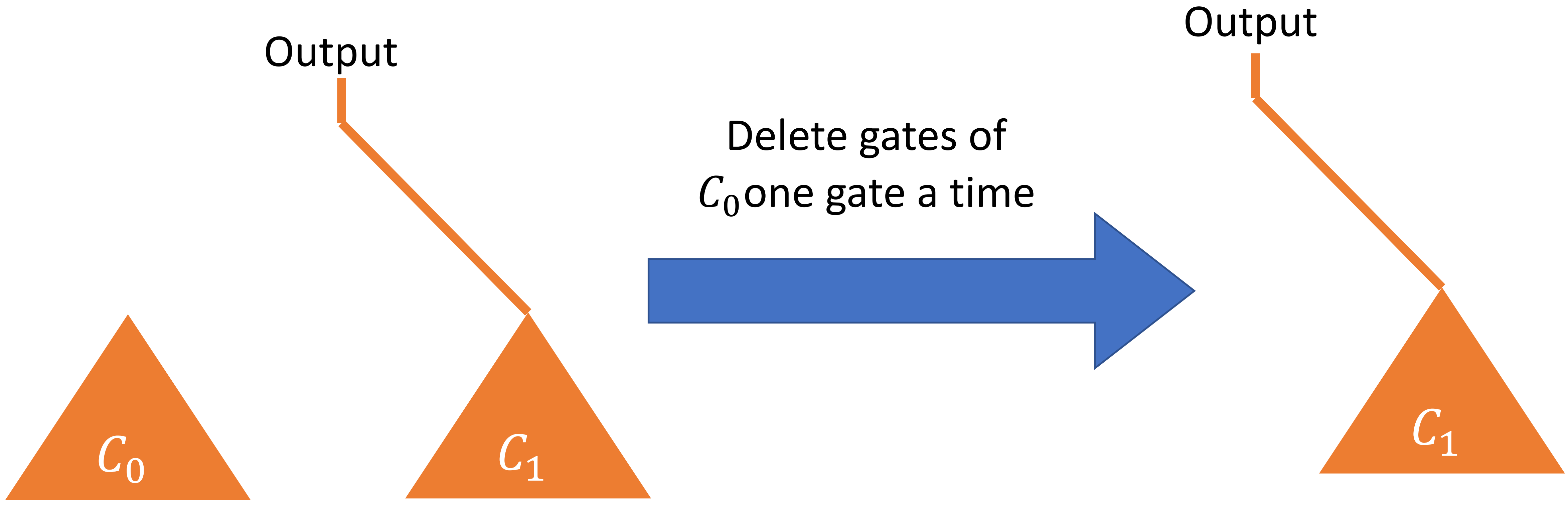

More Details (I): Multi-arity ∧-Gate?

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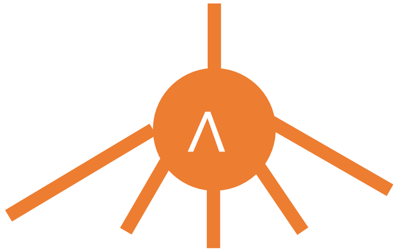

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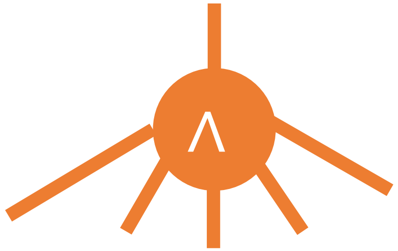

We need small arity, because

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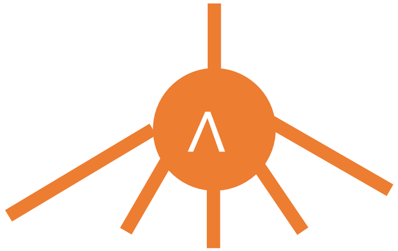
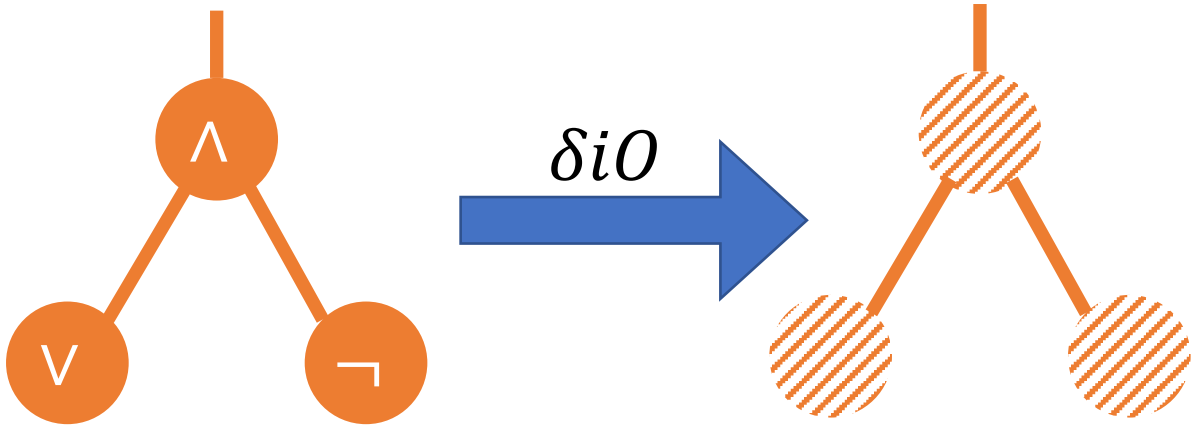
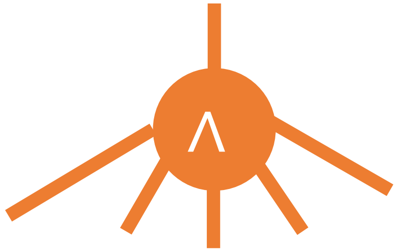


We need small arity, because

δiO relies on **iO for ckt** to obfuscate each gate

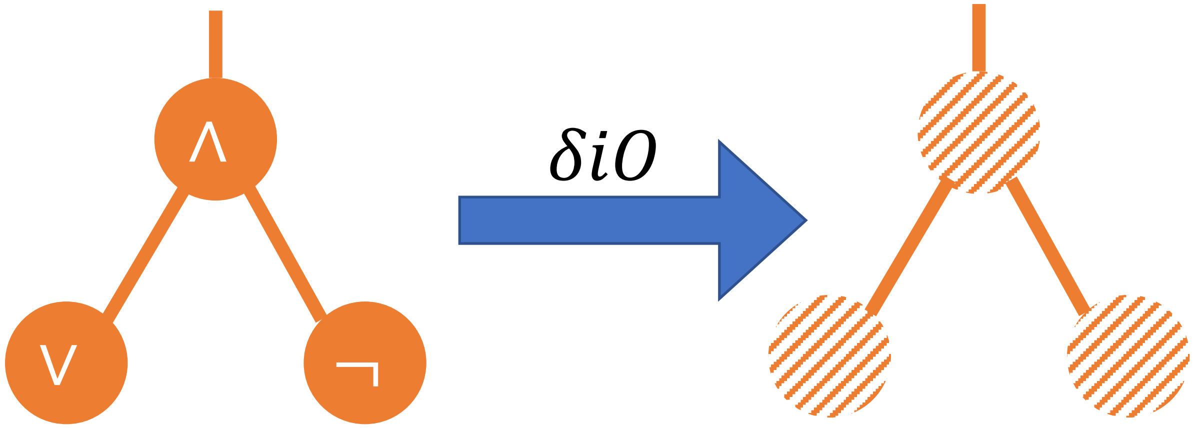


More Details (I): Multi-arity ∧-Gate?



We need small arity, because

δiO relies on **iO for ckt** to obfuscate each gate



Security Loss $\geq 2^{|arity|}$

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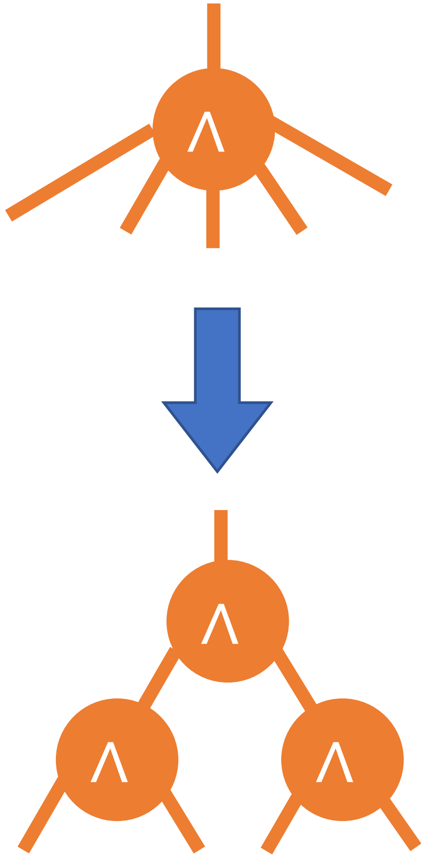

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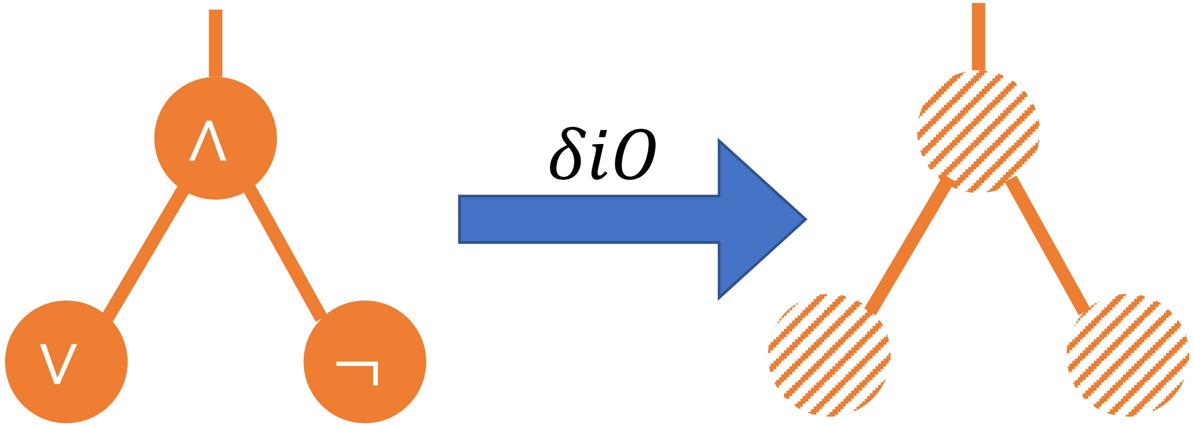

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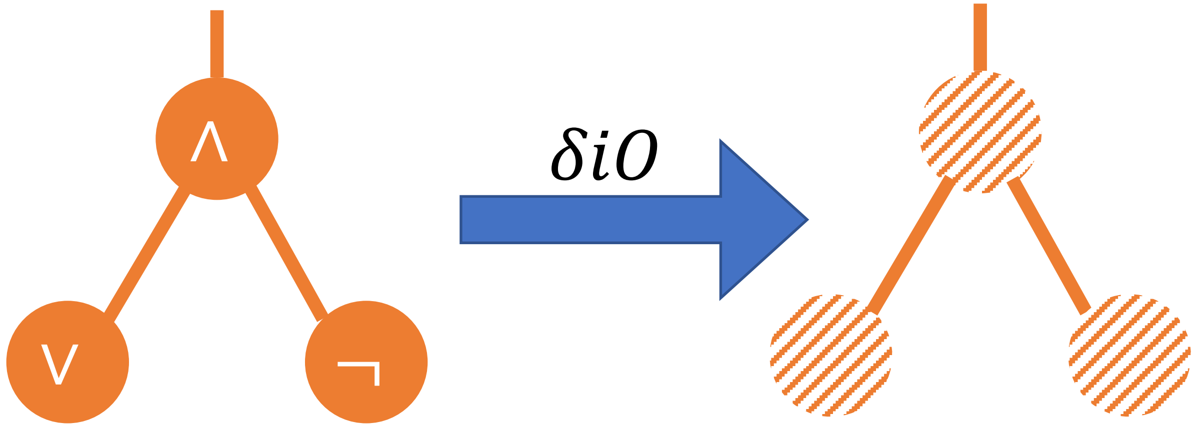

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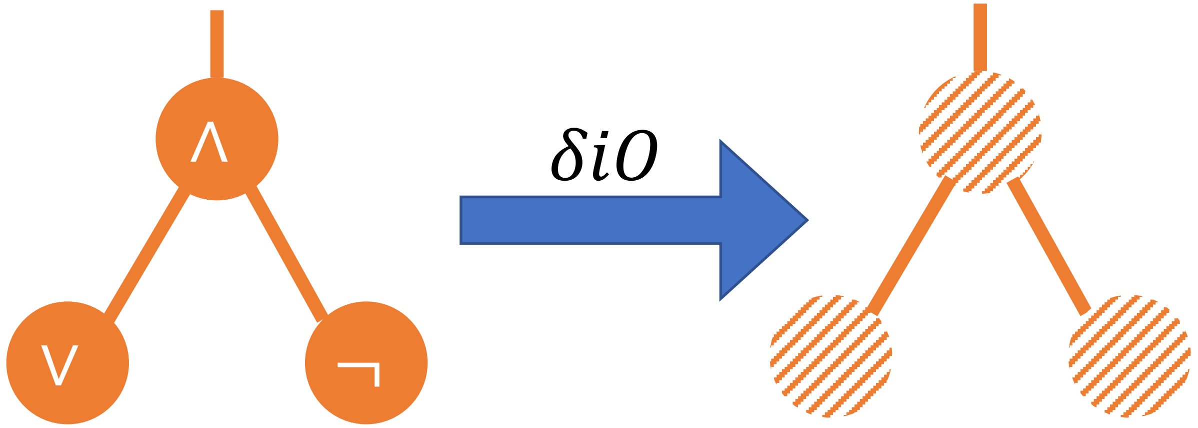

We can't **change** the **topology**, otherwise we can't apply the security of δiO.

Build “Helper” Sub-Circuits

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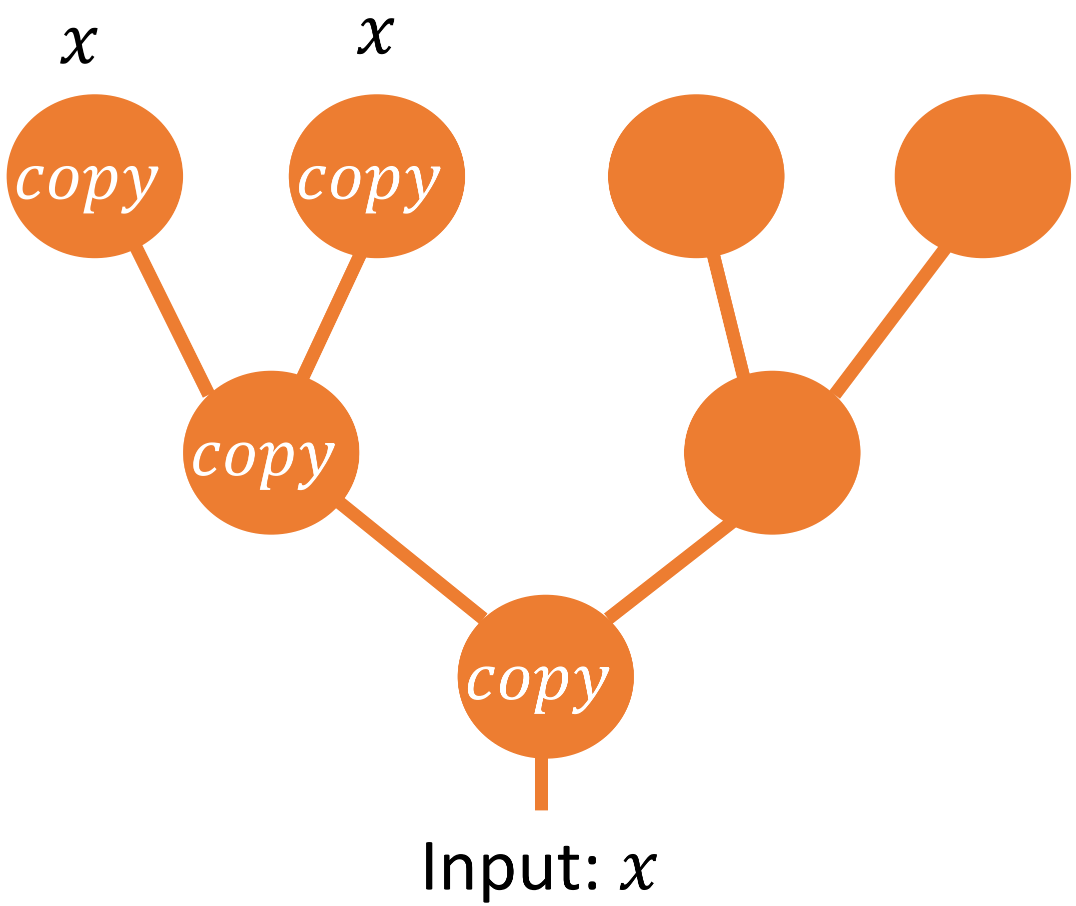

<u>"Copy" circuit</u>

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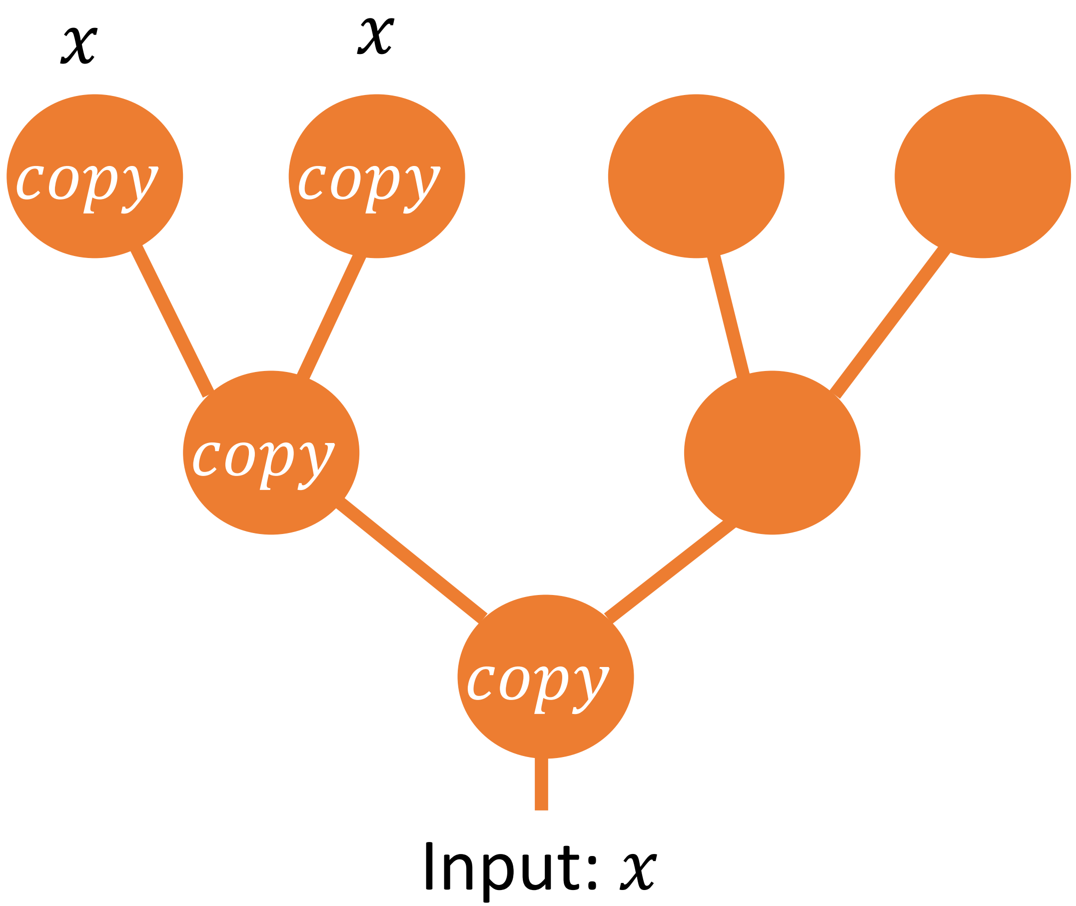

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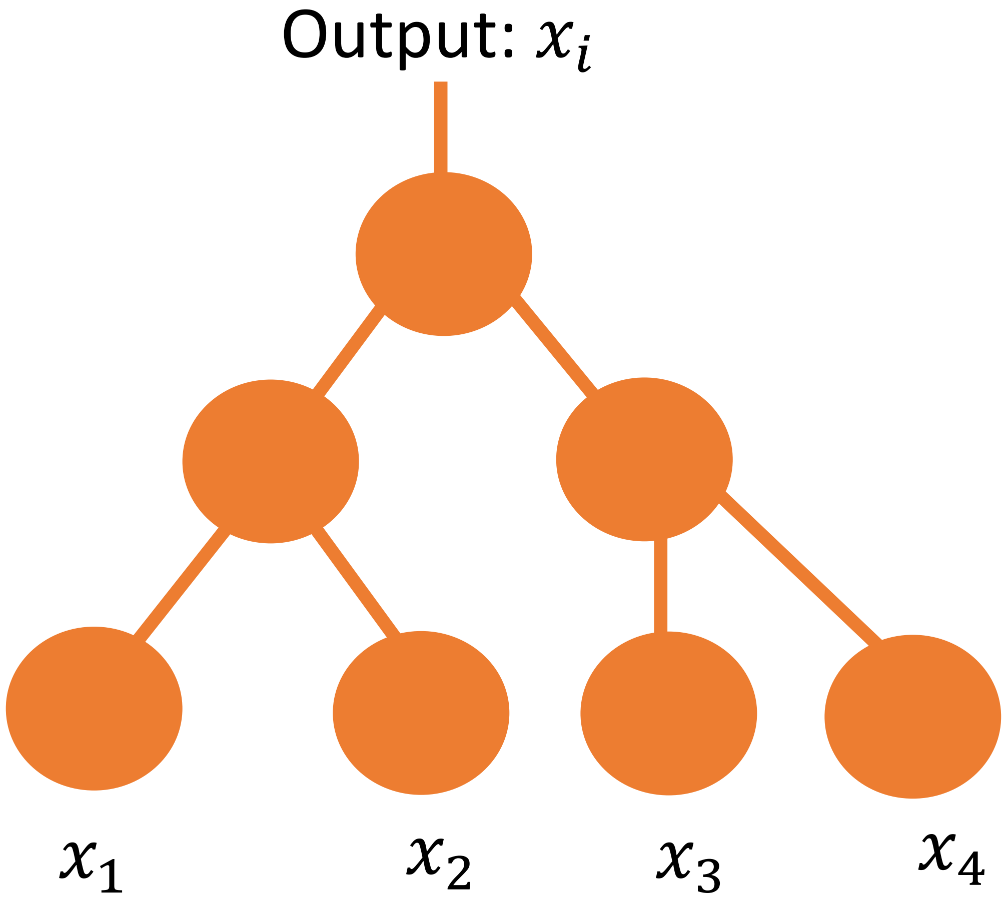
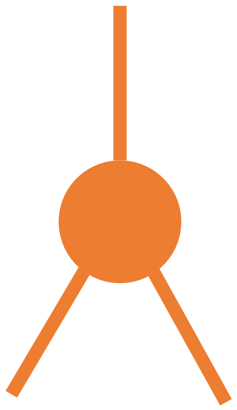


"Project" circuit

Pad the Circuit

Pad the Circuit



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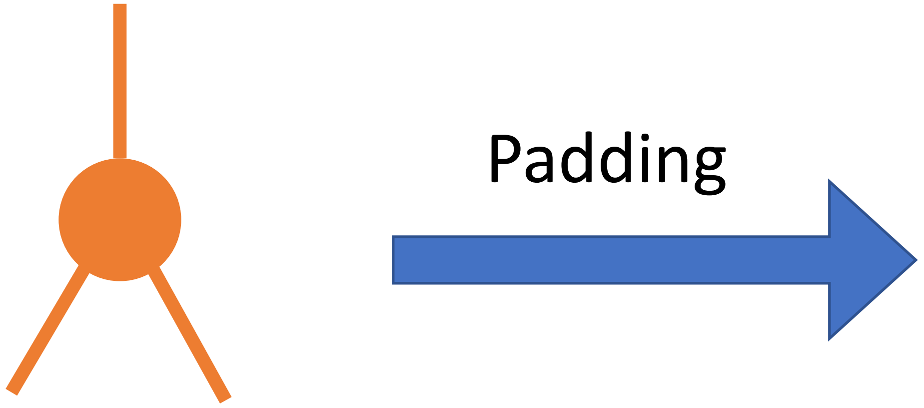

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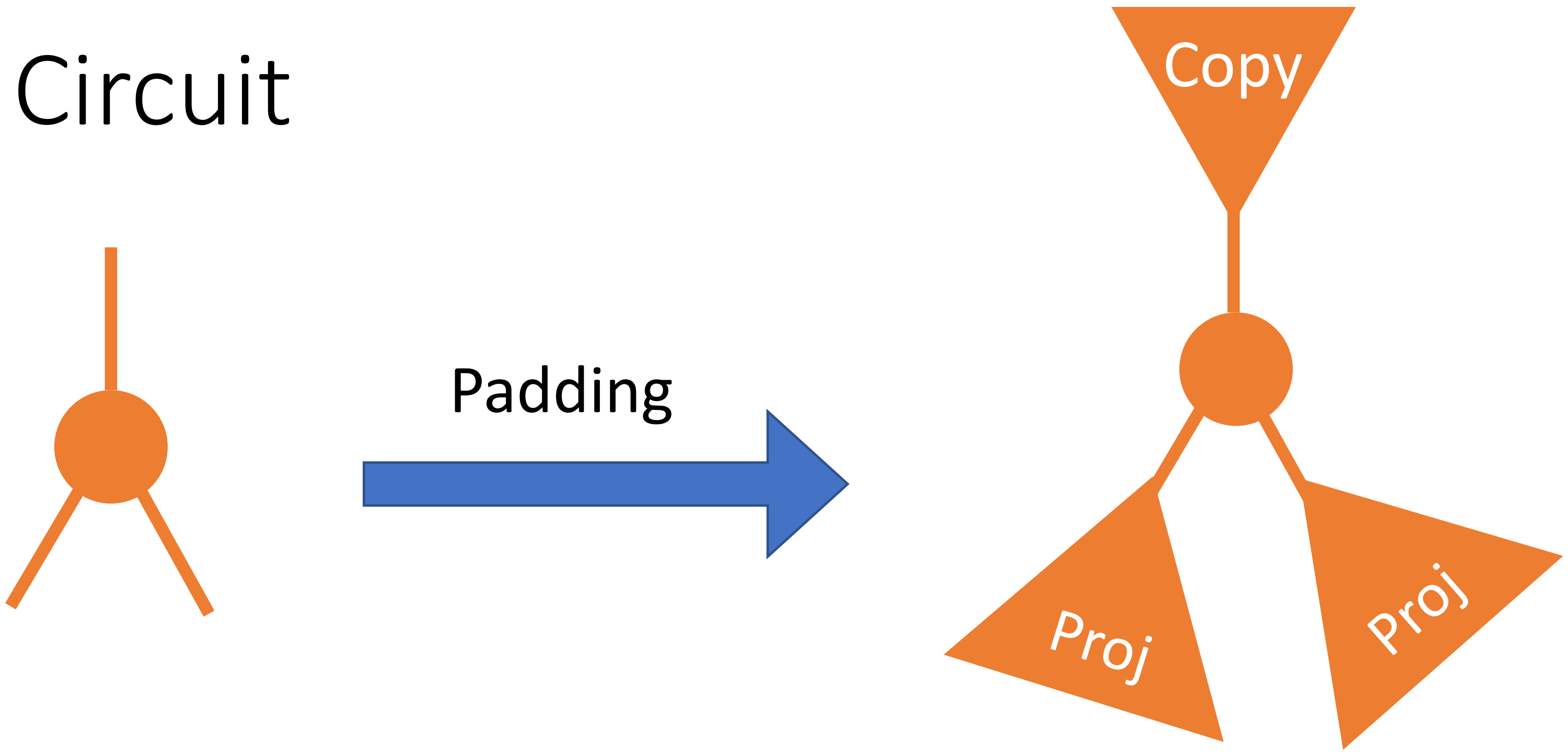

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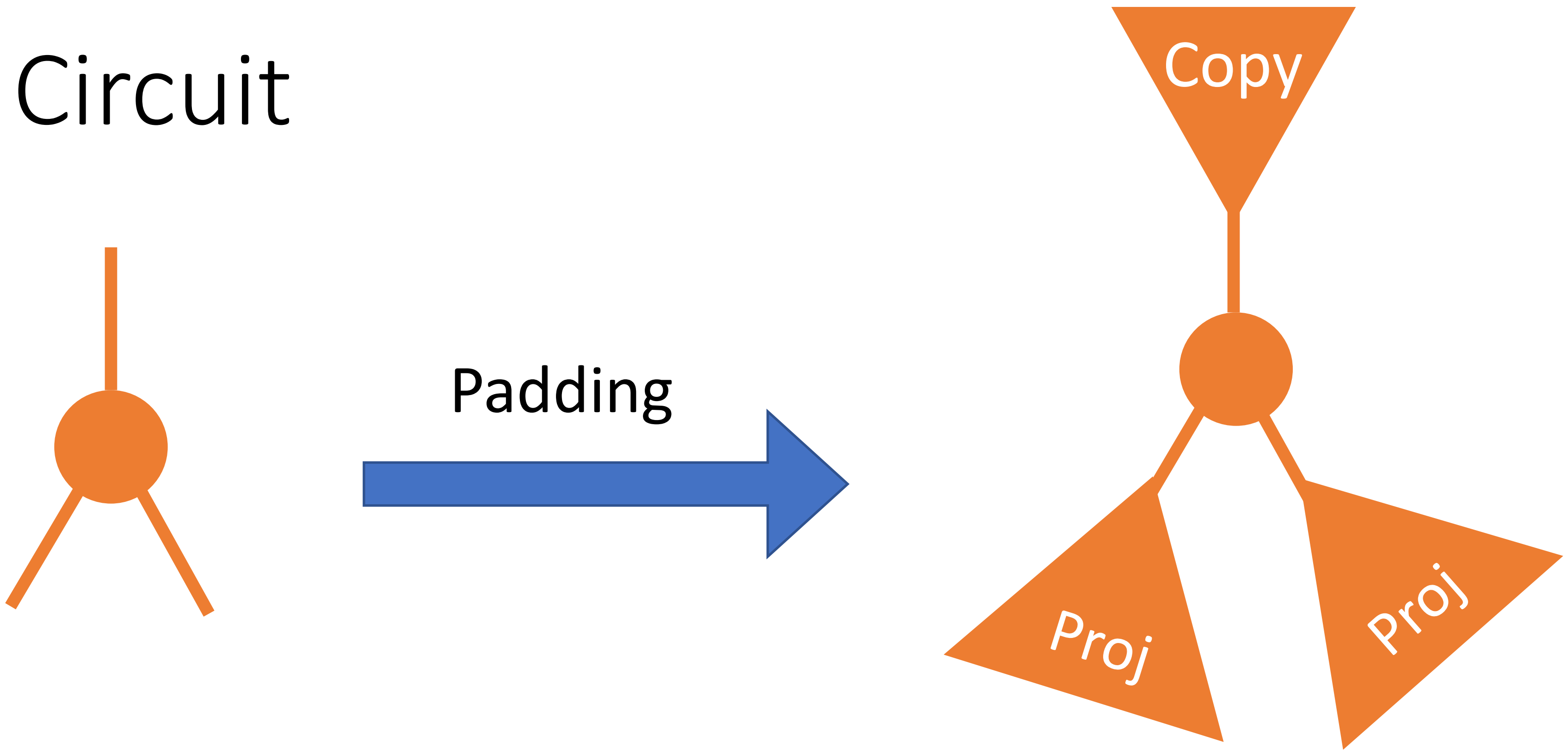

Topology changing operations (Adding/Deleting Gate) becomes changing the functionality of Copy, Proj sub-circuits.

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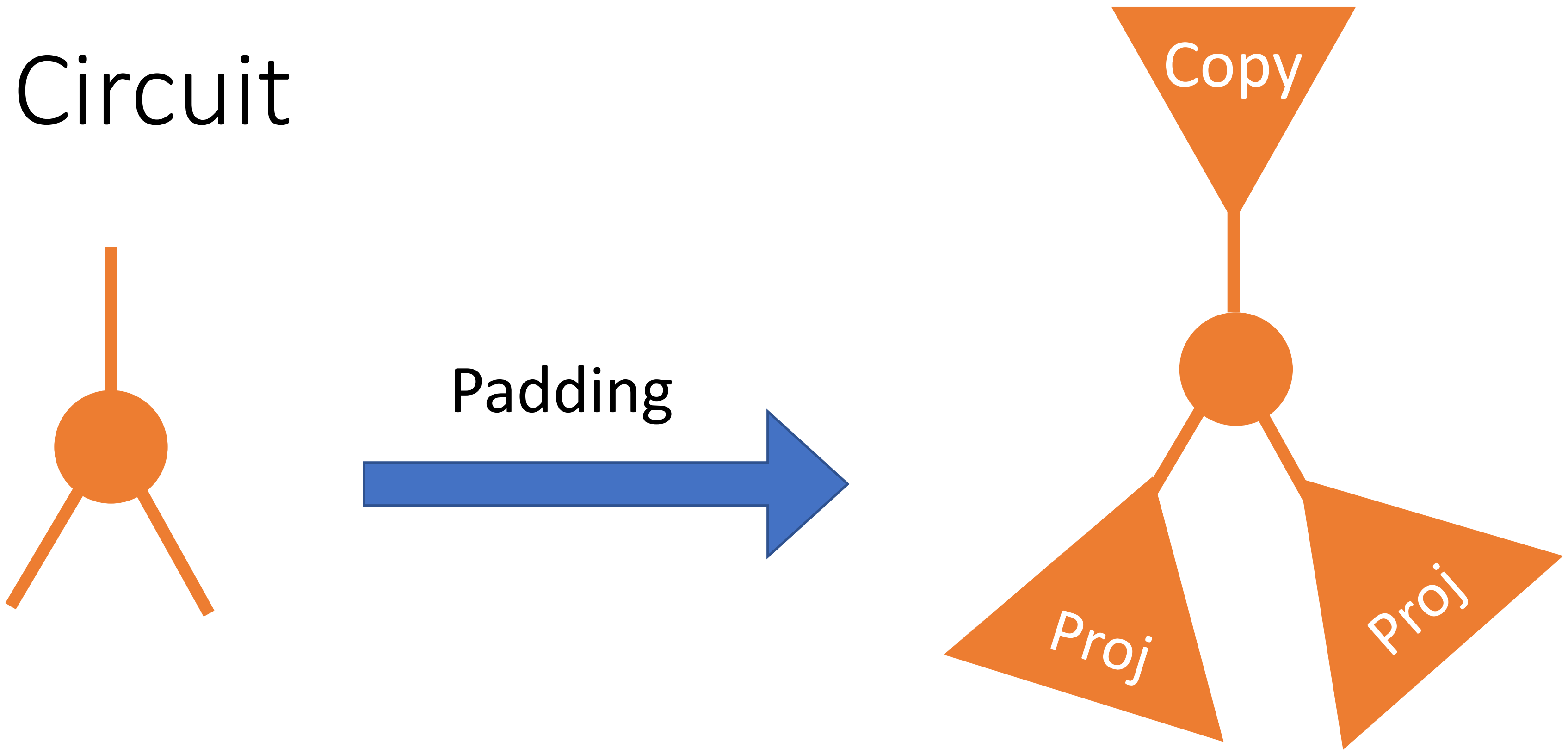

Topology changing operations (Adding/Deleting Gate) becomes changing the functionality of Copy, Proj sub-circuits.

The change is “local” due to the tree structure

Technical Details

- δ-Equivalence from $\mathcal{EF}$-proofs
- $\boldsymbol{\delta i\mathcal{O}}$ **Construction**
- iO for Turing Machines

Construct δiO: Initial Idea

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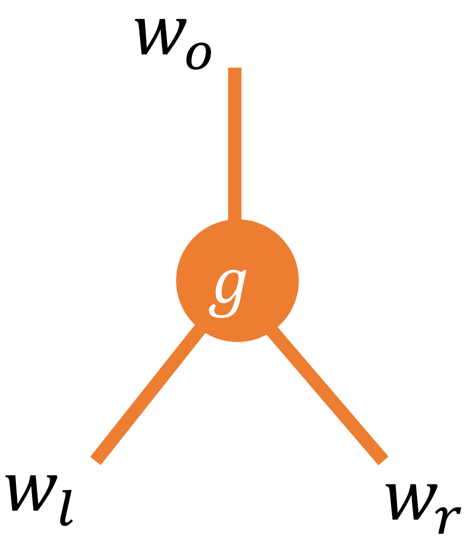

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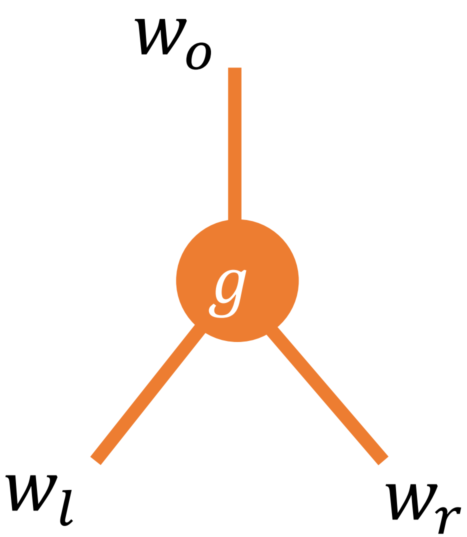

Use iO for Ckts to
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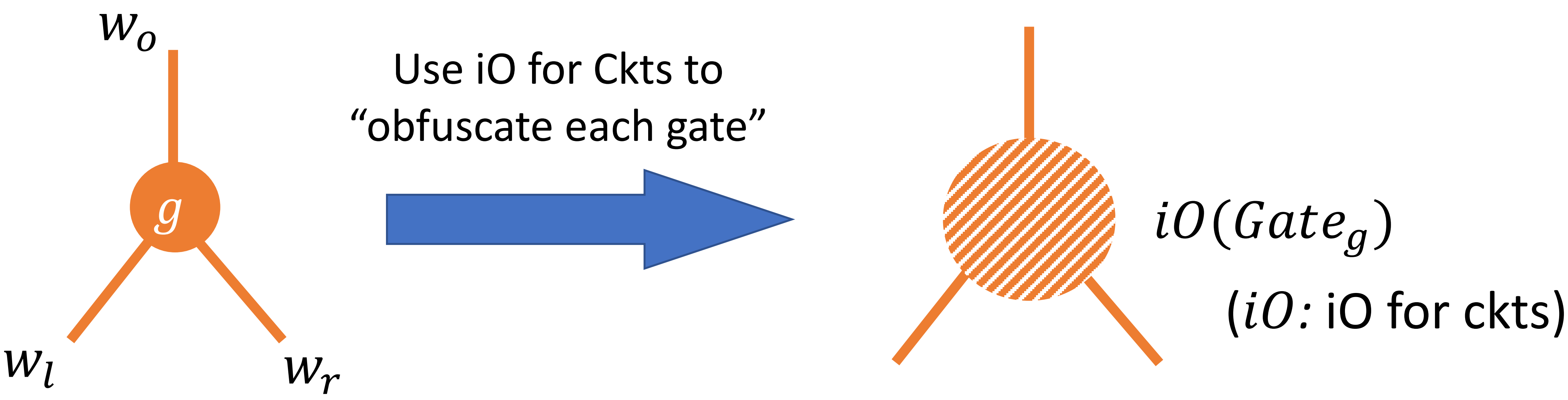

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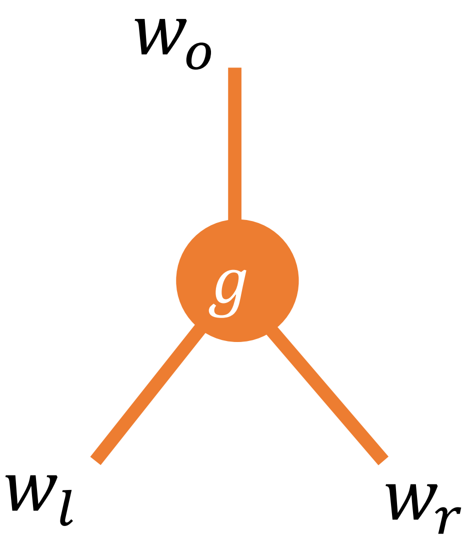

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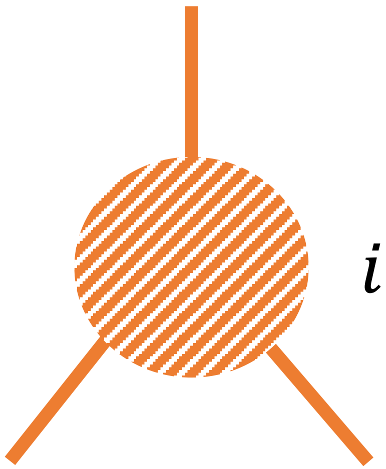

$iO(Gate_g)$

(iO: iO for ckts)

$Gate_g(w_l, w_r)$: Output $w_o = g(w_l, w_r)$

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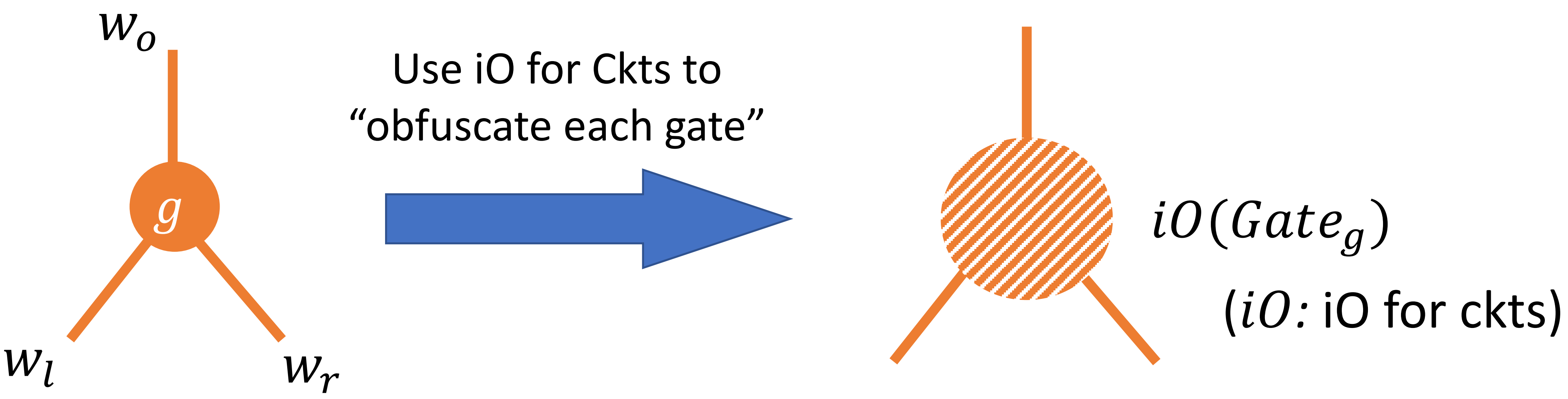

$Gate_g(w_l, w_r)$: Output $w_o = g(w_l, w_r)$

The adversary can learn the gate from its truth table.

Encrypt Wire Values

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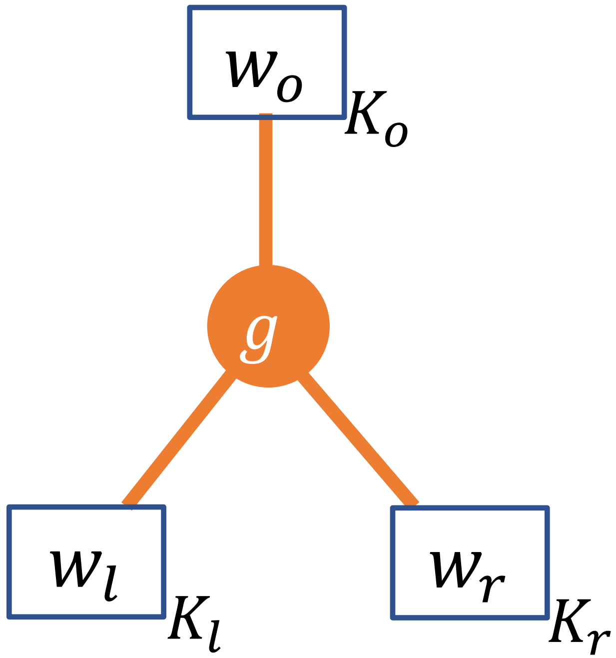

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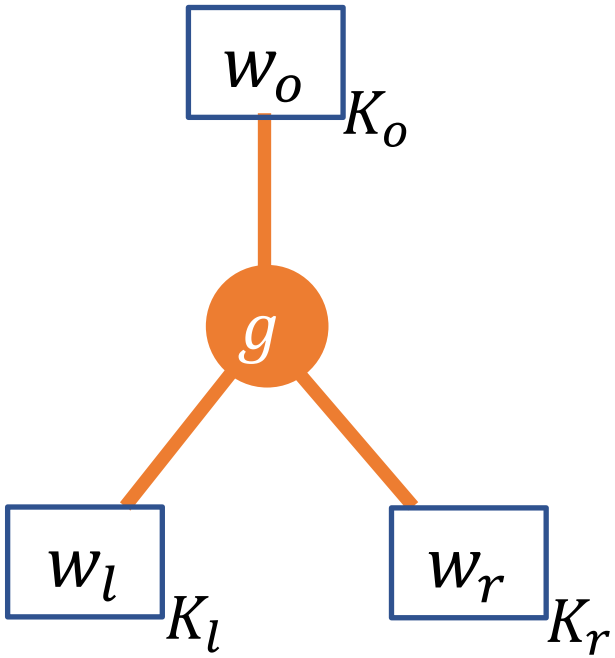
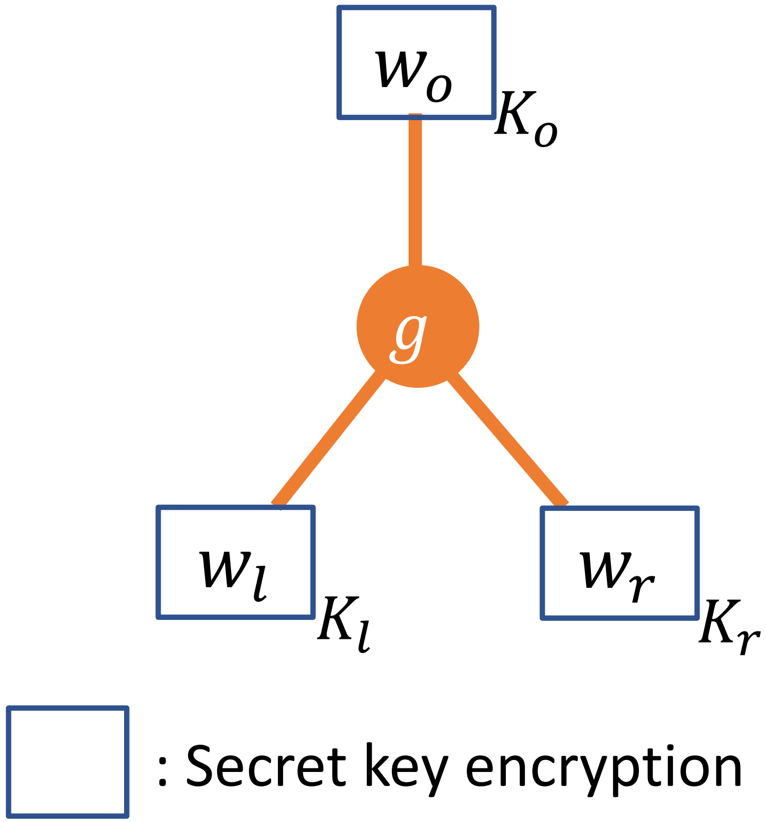


: Secret key encryption

Encrypt Wire Values



$\underline{Gate_g}$:

- **Input**: Ciphertexts of w_l, w_r

  Decrypt the input wires w_l, w_r
  Compute gate g: $w_o = g(w_l, w_r)$
  Encrypt the output wire w_o

- **Output**: Ciphertext of w_o,

Mix-and-Match Attack

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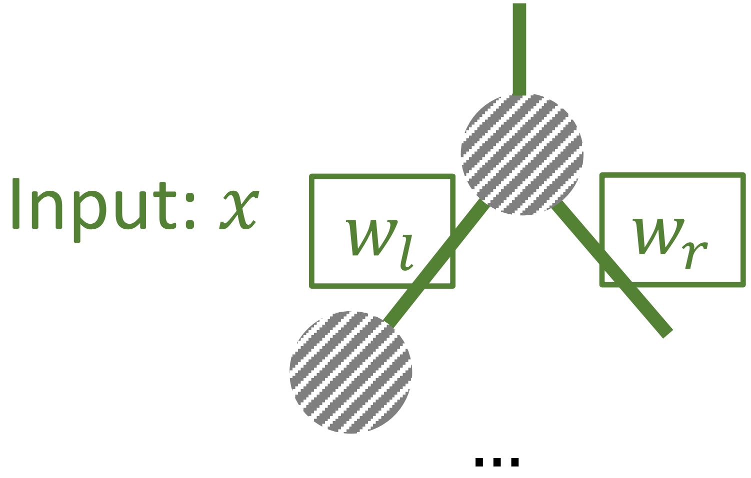

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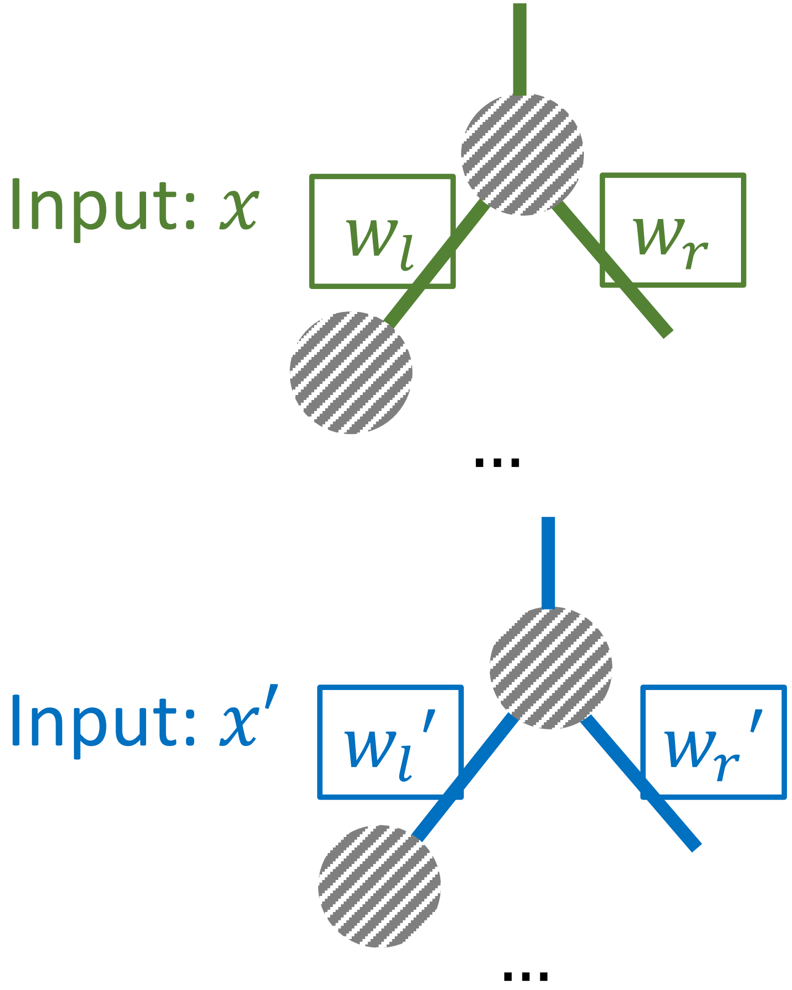

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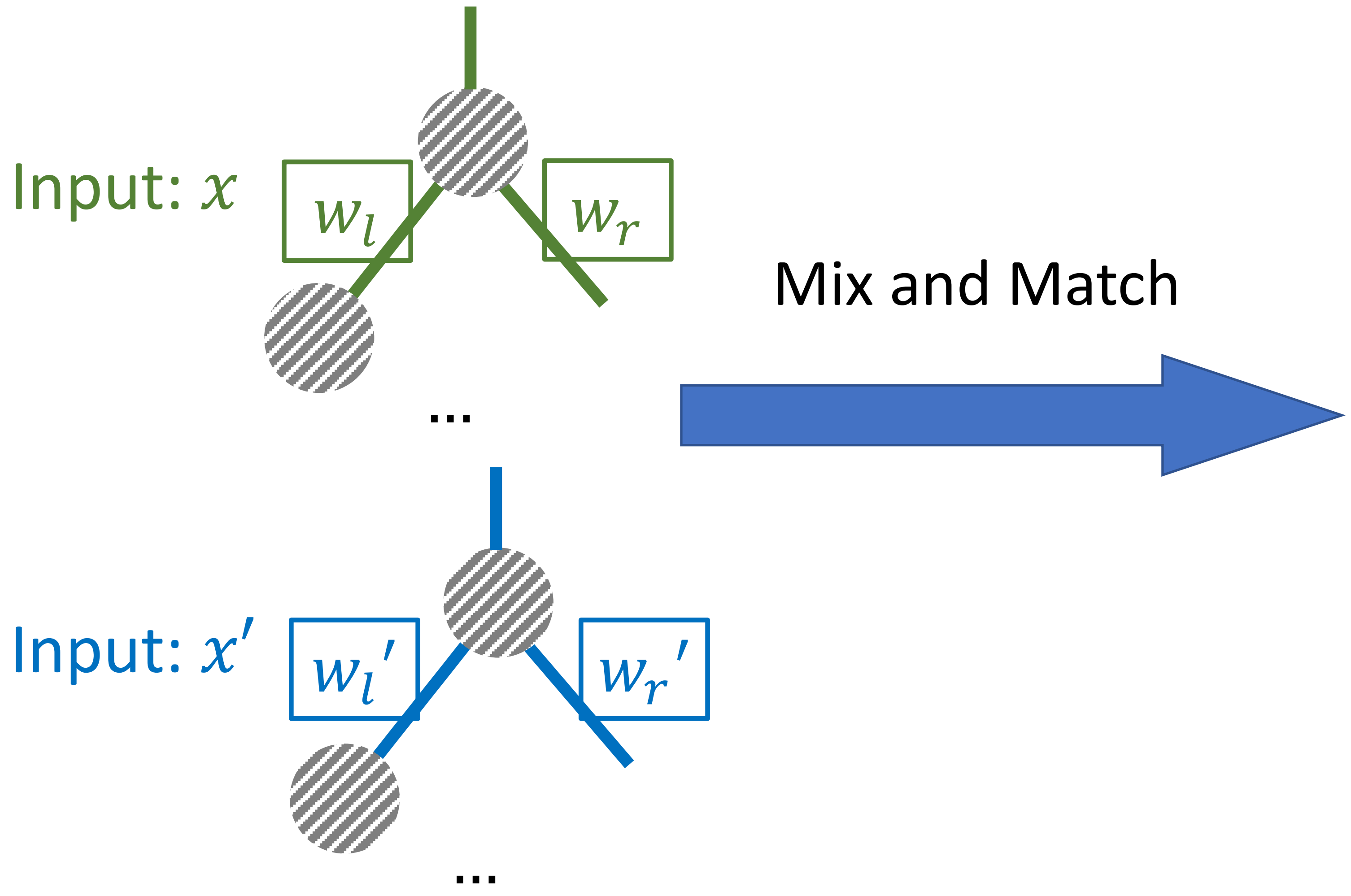

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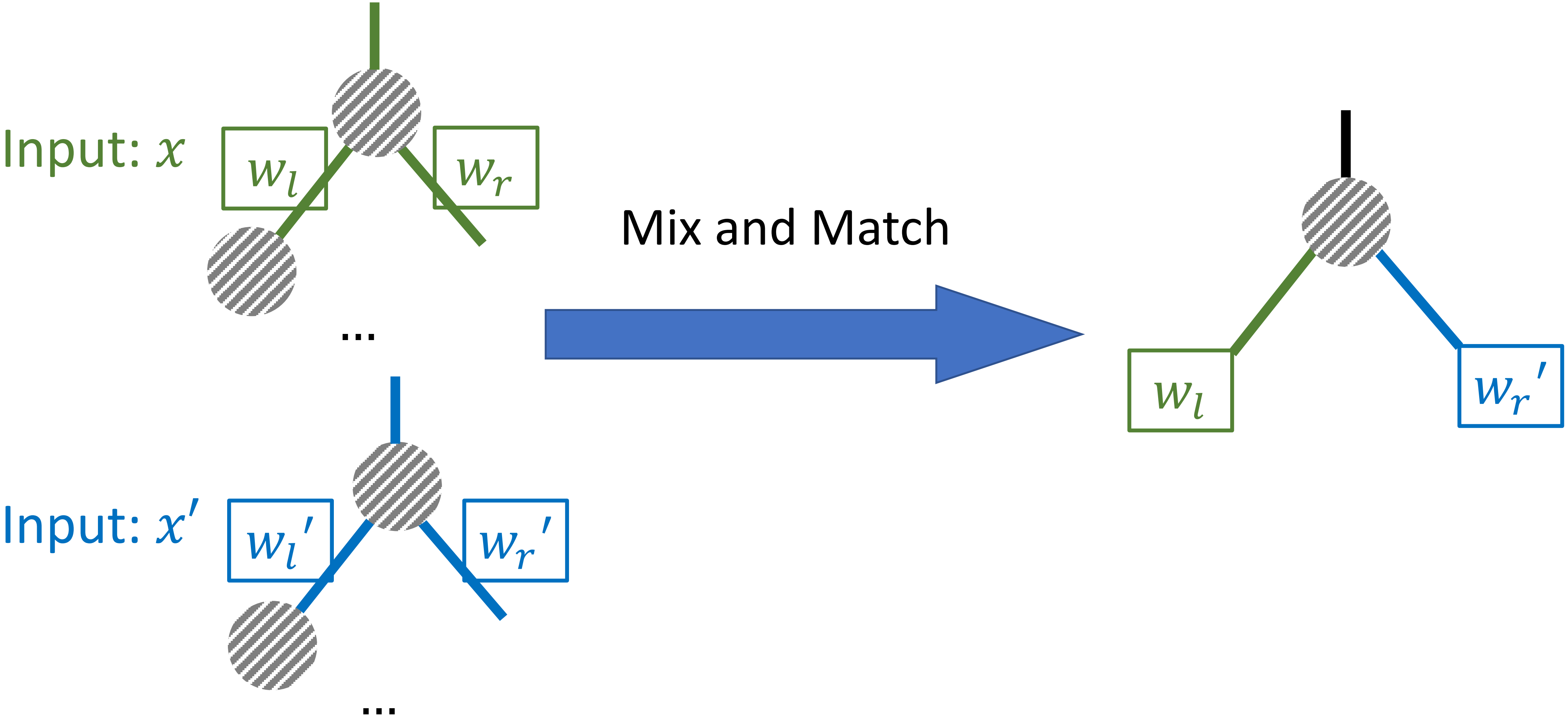

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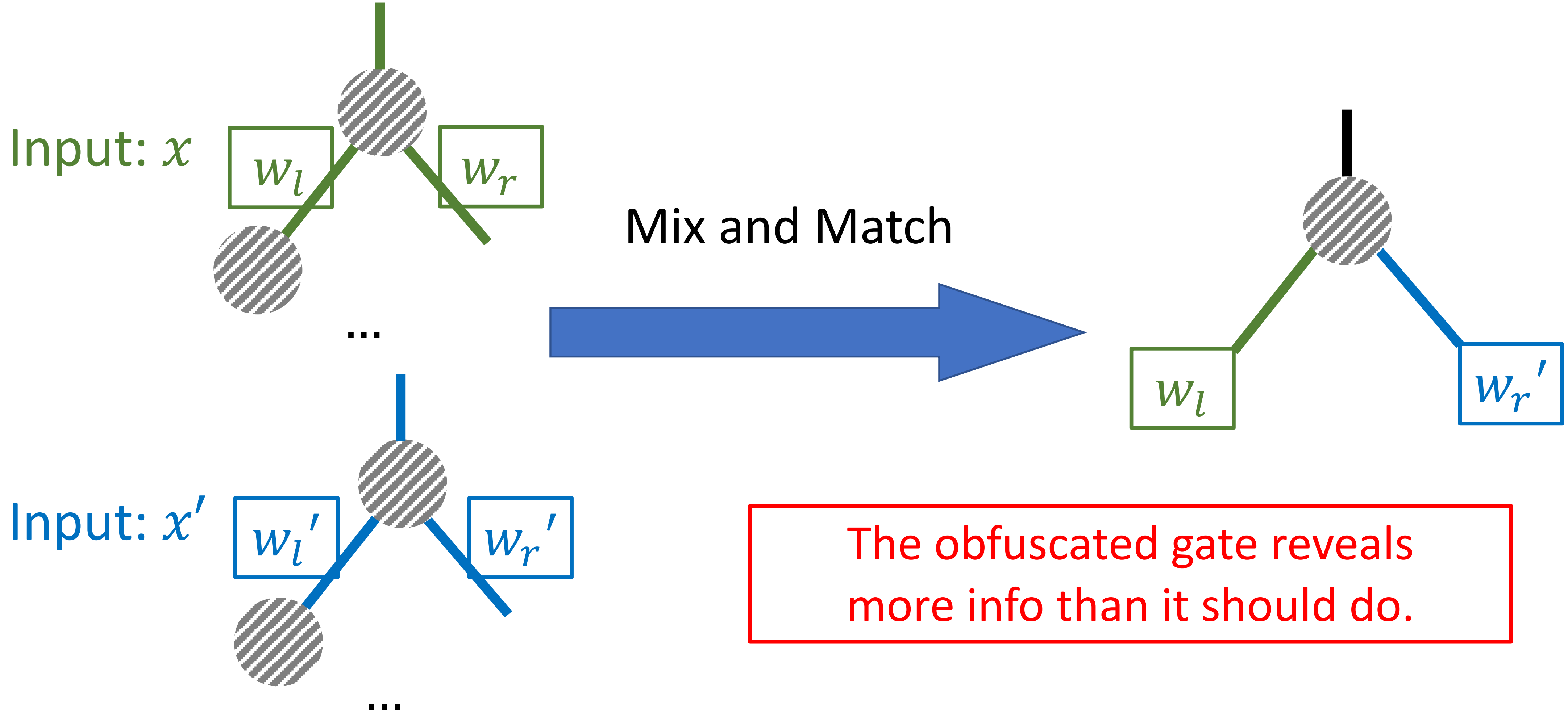

The obfuscated gate reveals more info than it should do.

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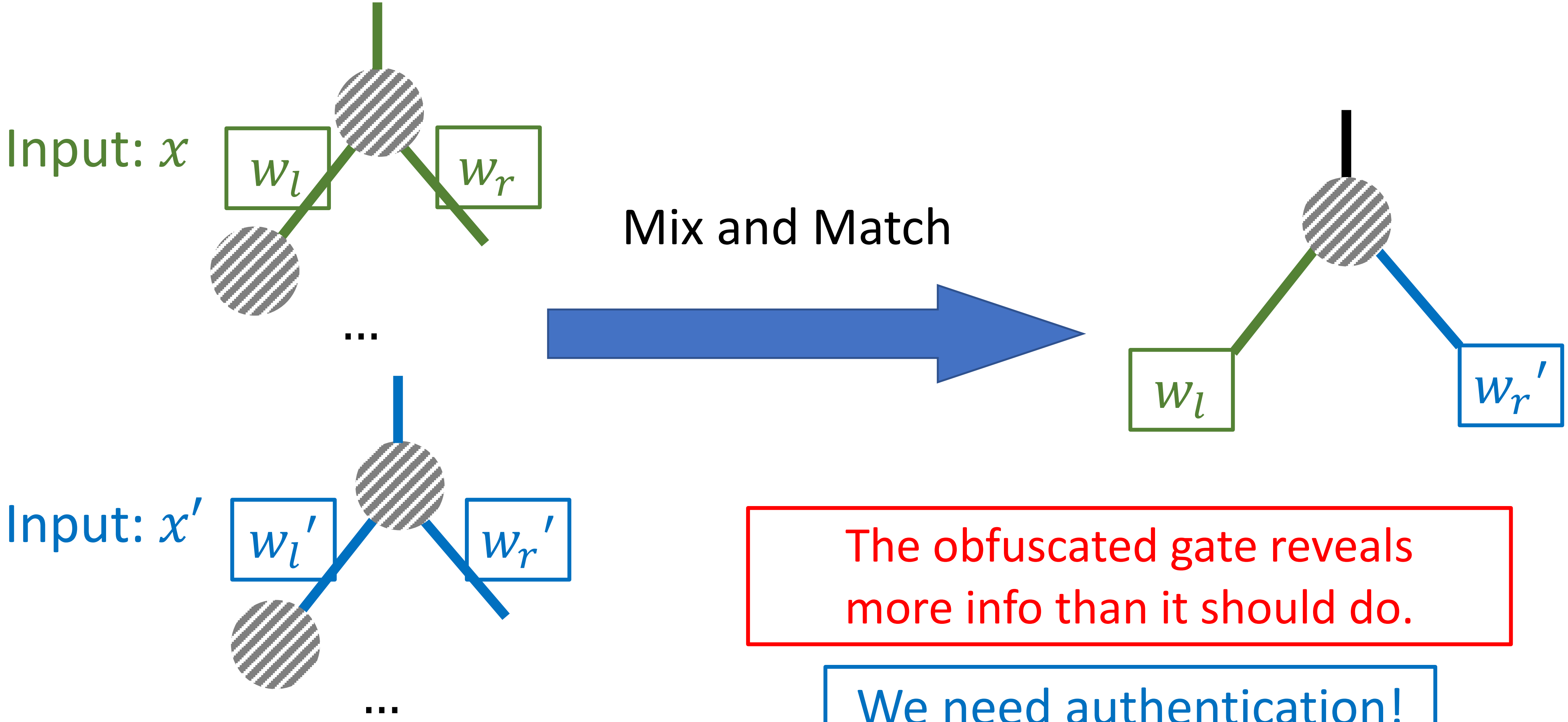

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We need authentication!

δiO Construction: Super High Level

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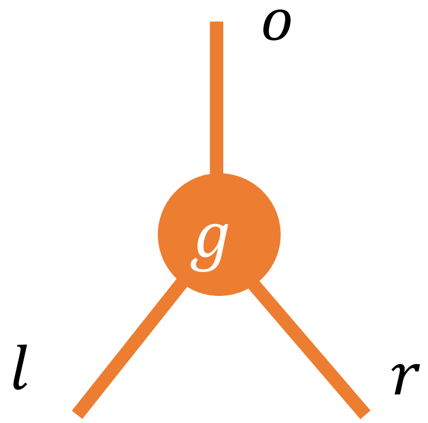

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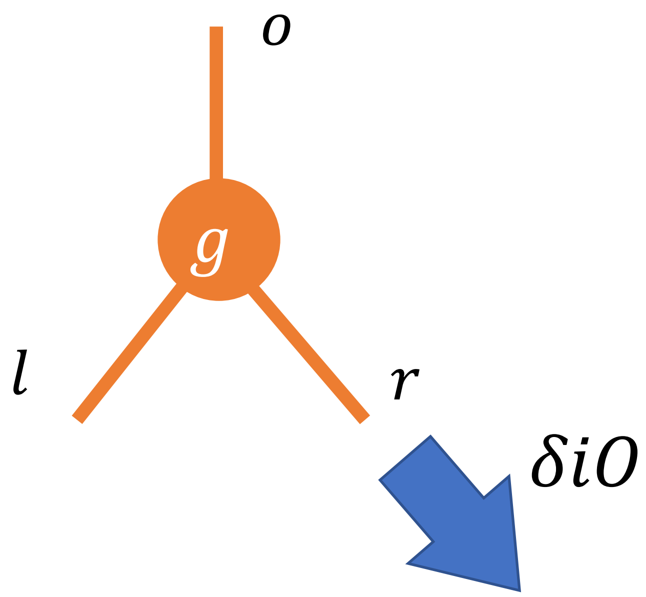

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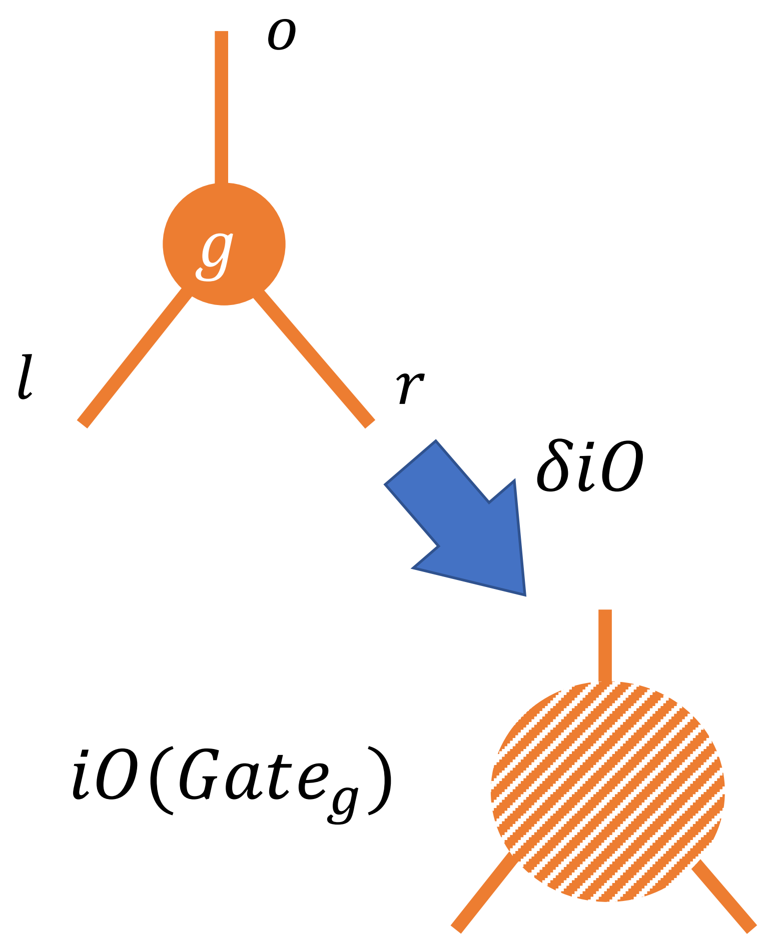

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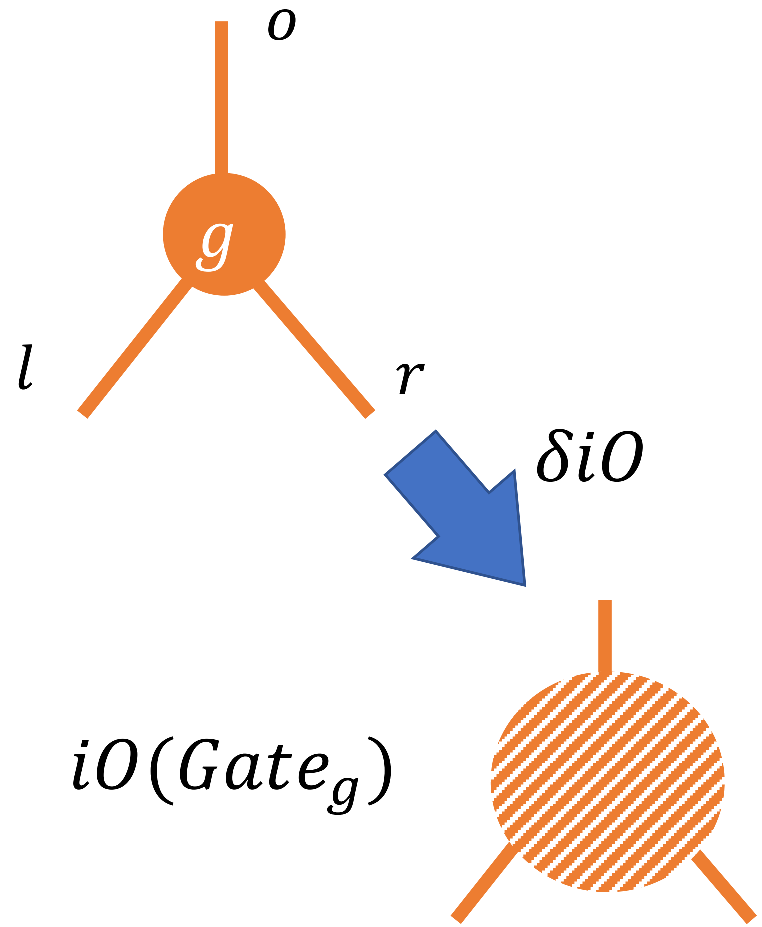

$\underline{Gate_g}$

Input: Ciphertext of input wire values,
Authentication info of l, r.

Verification of Authentication
Decrypt input wires
Compute gate g
Encrypt output wire

Output: Ciphertext of output wire
Authentication info of o.

Technical Details

- δ-Equivalence from $\mathcal{EF}$-proofs
- δiO Construction
- **iO for Turing Machines**

Recall: Cook's Translation

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Poly-size Proofs in $\mathcal{EF}$ $\xleftarrow{\text{Cook's Translation}}$ **Proofs in *PV***

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Poly-size Proofs in $\mathcal{EF}$ $\xleftarrow{\text{Cook's Translation}}$ **Proofs in *PV***

Fix Input length n

$\longleftarrow$ $\vdash_{PV} M_1(x) = M_2(x)$

Recall: Cook's Translation

Poly-size Proofs in $\mathcal{EF}$ $\longleftarrow$ Cook's Translation $\longleftarrow$ **Proofs in *PV***

Fix Input length n

$$\vdash_{\mathcal{EF}} C_{M_1,n}(x) \leftrightarrow C_{M_2,n}(x) \longleftarrow \vdash_{PV} M_1(x) = M_2(x)$$

$C_{M_i,n}(x)$: Circuit that computes M_i for input $|x| \leq n$.

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Poly-size Proofs in $\mathcal{EF}$ $\xleftarrow{\text{Cook's Translation}}$ **Proofs in *PV***

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We know how to build iO for circuits
of poly-size $\mathcal{EF}$-proof of equivalence : δiO

iO for TMs from δiO

iO for TMs from δiO

M

Turing Machine

iO for TMs from δiO

$(N_0 = \lambda^{\log \lambda})$

M

Turing Machine $\xrightarrow[\;]{\text{All Input length } n \le N_0}$

iO for TMs from δiO

$(N_0 = \lambda^{\log \lambda})$

All Input length
$n \leq N_0$

M
Turing Machine $\xrightarrow{\hspace{2em}}$

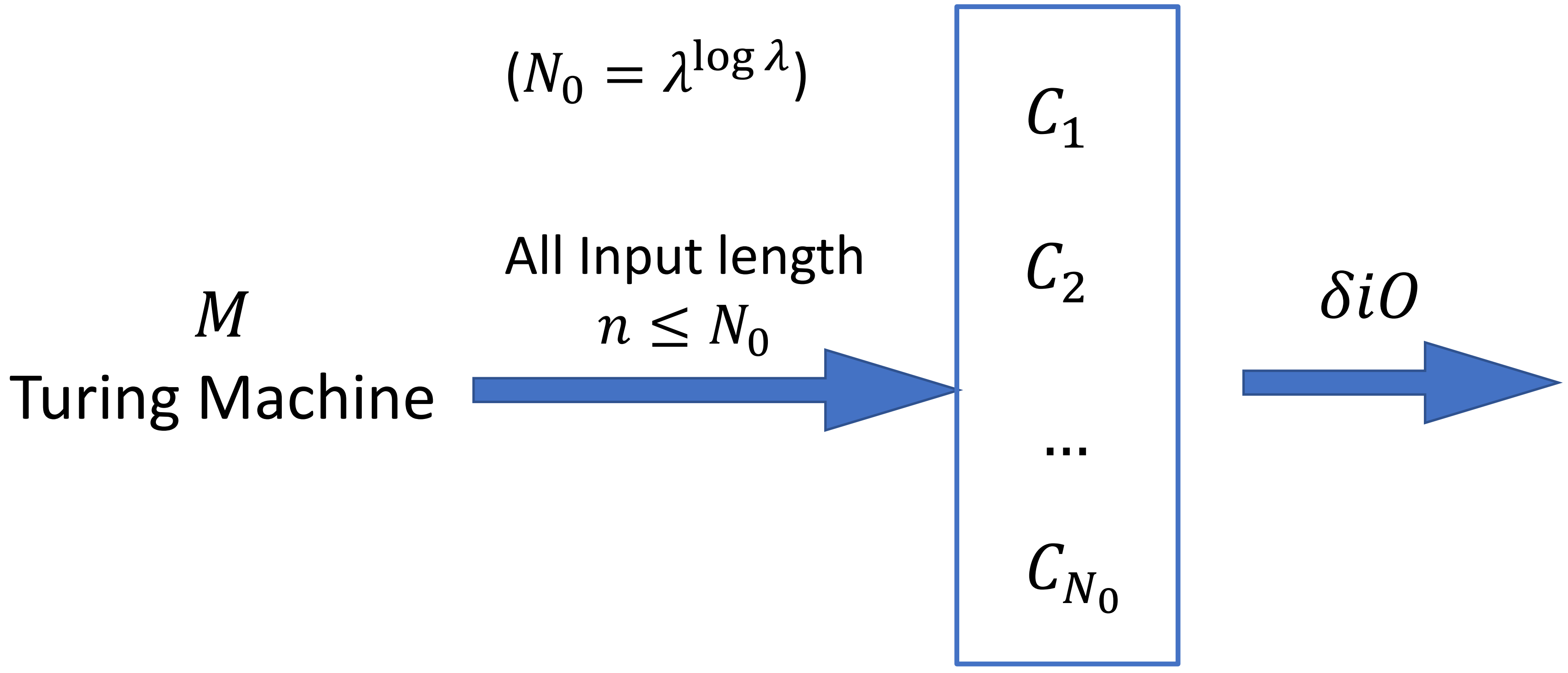
C_1

C_2

...

C_{N_0}

iO for TMs from δiO



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C_1

C_2

...

C_{N_0}

δiO

$\delta iO(C_1)$

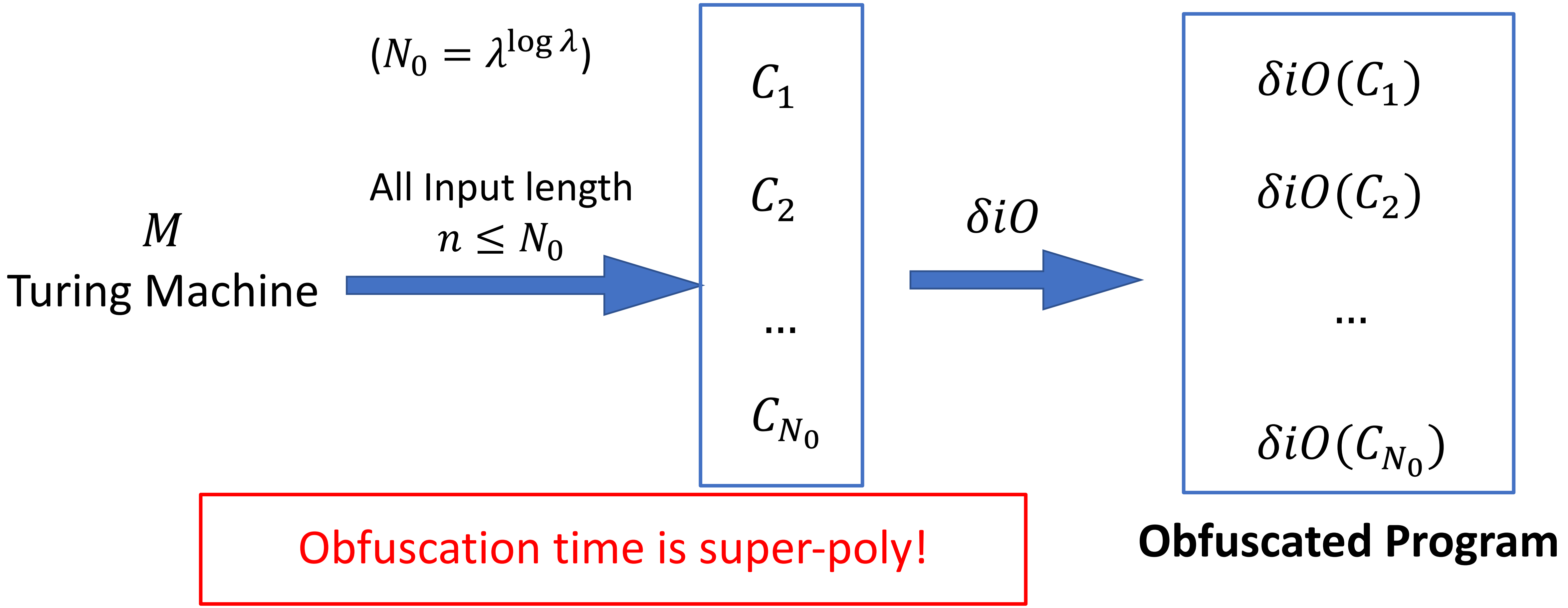
$\delta iO(C_2)$

...

$\delta iO(C_{N_0})$

Obfuscated Program

iO for TMs from δiO



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$C_1, C_2, \dots, C_{N_0}$ have a succinct description

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Recall: δiO Construction

Gates $g_1, g_2, \ldots, g_{|C|}$ $\xrightarrow{\delta iO}$

$iO(Gate_{g_1})$

$iO(Gate_{g_2})$

…

$iO(Gate_{g_{|C|}})$

Generate $Gate_g$ from $[M](\cdot,\cdot)$

Efficient Construction

Efficient Construction

M

Turing Machine

Efficient Construction

M
Turing Machine

Efficient Construction

M
Turing Machine

$$\Rightarrow$$

$UGate(\cdot,\cdot,\cdot)$

Efficient Construction

(iO for small circuit)

M
Turing Machine $\Rightarrow$ $UGate(\cdot,\cdot,\cdot)$ $\Rightarrow$ $iO(UGate)$

Efficient Construction

(iO for small circuit)

M
Turing Machine → $UGate(\cdot,\cdot,\cdot)$ → $iO(UGate)$

"Uniform" Gate $UGate(n, i, input')$

Get description of i-th gate:

$$g \leftarrow [M](n, i)$$

Compute and output

$$Gate_g(input')$$

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  Unprovability of cryptographic problems?